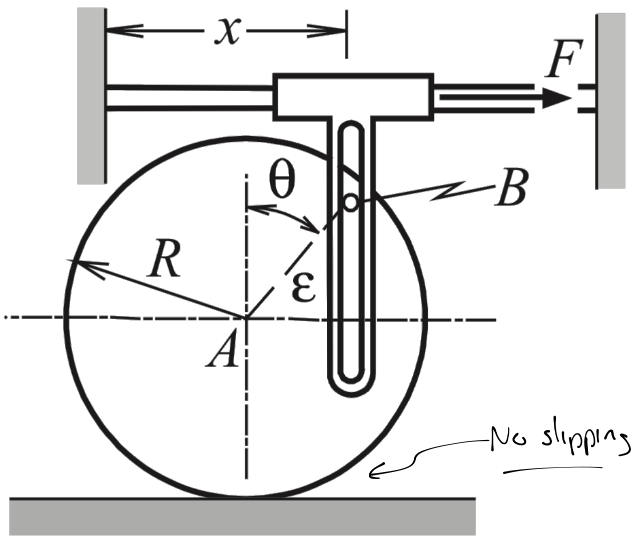
Example 8.3

EXAMPLE 8.3 A known force $\bar{F}$ is applied to the yoke, causing the system to move rightward. The disk rolls without slipping, and the coefficient of kinetic friction between pin B and the yoke is μ. Masses are m_1 for the yoke and m_2 for the homogeneous disk. Derive the equations of motion.



Q: How many DoF are there?

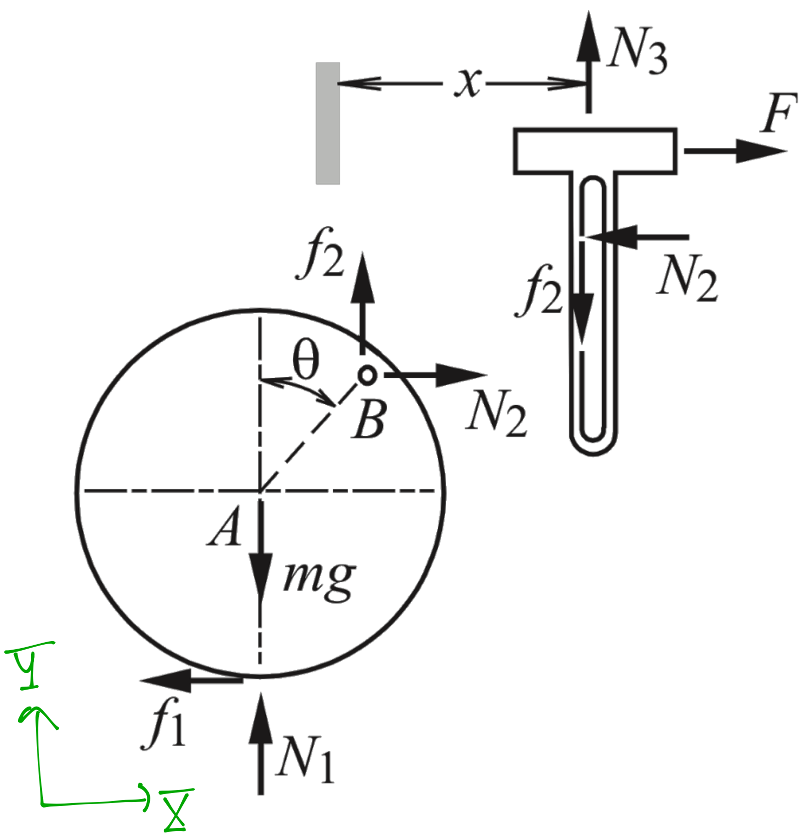
only 1 due to no slipping at the ground.

Selecting x and θ independently would require slipping

But, we need to include both in order to determine the effects of friction in the slot.

Choose: $\bar{q} = (\theta, x)$

In doing this, we must also define a constraint that relates these two coordinates



Look at point B.

$\bar{V}_B = \dot{x}\bar{I}$ and $\bar{V}_B = \bar{V}_A + \bar{\omega} \times \bar{r}_{B/A}$ ← Both must be true

$\bar{V}_A = R\dot{\theta}\bar{I}$ (due to rolling)

$\bar{\omega} = -\dot{\theta}\bar{K}$ $\bar{r}_{B/A} = \varepsilon\sin\theta\,\bar{I} + \varepsilon\cos\theta\,\bar{J}$

So

$$\dot{x}\bar{I} = R\dot{\theta}\bar{I} + (-\dot{\theta}\bar{K}) \times (\varepsilon\sin\theta\,\bar{I} + \varepsilon\cos\theta\,\bar{J})$$

$$\underbrace{\dot{x}\bar{I} = (R\dot{\theta} + \varepsilon\dot{\theta}\cos\theta)\bar{I}} + (-\varepsilon\dot{\theta}\sin\theta\,\bar{J})$$

$$\dot{x} - (R + \varepsilon\cos\theta)\dot{\theta} = 0$$

Putting into "normal" constraint form

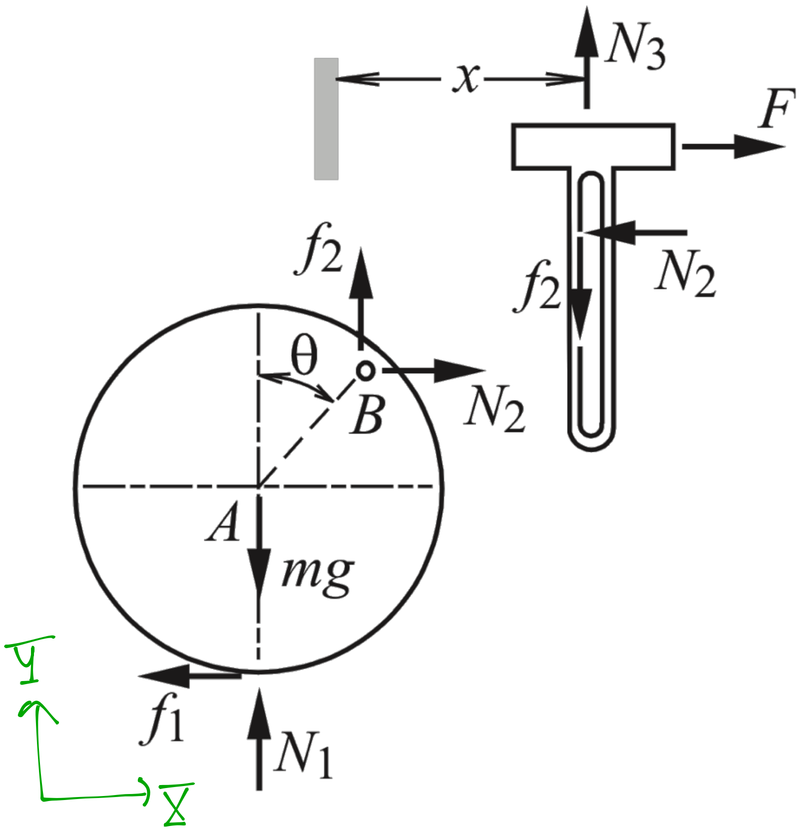
$$\dot{x} - (R + \varepsilon\cos\theta)\dot{\theta} = 0 \longleftrightarrow a_{11}\dot{x} + a_{12}\dot{\theta} + b_1 = 0 \longrightarrow a_{11} = 1 \quad a_{12} = -(R + \varepsilon\cos\theta) \quad b_1 = 0$$

Q: We don't need to include N_1, f_1, or N_3 in the formulation. Why?

f_1 and N_1 are constraint forces that enforce no slip — not violated by x and θ change

N_3 is constraint on vertical location of collar - likewise not violated so no virtual work

Example 8.3 (cont.)



Forces f_2 and N_2 do virtual work on both the yoke and the disk. So, we need to write virtual displacements for the point of application for both

$$\delta \bar{r}_{yoke} = \delta x \bar{\imath} \quad \leftarrow \text{yoke moves only in } \bar{I} \text{ by } \delta x$$

To find $\delta \bar{r}_B$ use kinematical method. We already found that:

$$\bar{v}_B = (R\dot{\theta} + \varepsilon\dot{\theta}\cos\theta)\bar{I} + (-\varepsilon\dot{\theta}\sin\theta\bar{J})$$
$$= (R + \varepsilon\cos\theta)\dot{\theta}\bar{I} + (-\varepsilon\sin\theta)\dot{\theta}\bar{J}$$

So

$$\delta\bar{r}_B = (R + \varepsilon\cos\theta)\delta\theta\bar{I} + (-\varepsilon\sin\theta)\delta\theta\bar{J}$$

Now, we can write the virtual work equation

$$\delta W = (N_2\bar{I} + f_2\bar{J})\cdot\delta\bar{r}_B + ((F - N_2)\bar{I} - f_2\bar{J})\cdot\delta\bar{r}_{yoke}$$
$$= N_2(R + \varepsilon\cos\theta)\delta\theta - f_2\varepsilon\sin\theta\delta\theta + (F - N_2)\delta x$$
$$= \underbrace{[N_2(R + \varepsilon\cos\theta) - f_2\varepsilon\sin\theta]}_{Q_1}\delta\theta + \underbrace{[F - N_2]}_{Q_2}\delta x \quad = Q_1\delta\theta + Q_2\delta x$$

We now need to form the Lagrangian

$$T = \underbrace{\tfrac{1}{2}m_1\bar{v}_A\cdot\bar{v}_A + \tfrac{1}{2}I_{zz}\dot{\theta}^2}_{\text{Disk}} + \underbrace{\tfrac{1}{2}m_2\dot{x}^2}_{\text{yoke only translates}}$$

I_{zz}: disk moment of inertia about $A = \frac{1}{2}m_1R^2$

$$T = \tfrac{1}{2}m_1(R\dot{\theta})^2 + \tfrac{1}{2}\left(\tfrac{1}{2}m_1R^2\right)\dot{\theta}^2 + \tfrac{1}{2}m_2\dot{x}^2$$
$$= \tfrac{1}{2}\left(\tfrac{3}{2}m_1R^2\right)\dot{\theta}^2 + \tfrac{1}{2}m_2\dot{x}^2$$

$V = 0$

So $L = \frac{1}{2}\left(\frac{3}{2}m_1R^2\right)\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2$

Example 8.3 (cont.)

$$L = \frac{1}{2}\left(\frac{3}{2}m_1R^2\right)\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 \qquad Q_1 = N_2\left(R+\varepsilon\cos\theta\right) - f_2\,\varepsilon\sin\theta \quad \text{and} \quad Q_2 = F - N_2$$

For $q_1 = \theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = Q_1 \qquad \frac{\partial L}{\partial\dot{\theta}} = \frac{3}{2}m_1R^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{3}{2}m_1R^2\ddot{\theta} \qquad \frac{\partial L}{\partial\theta} = 0$$

$$\frac{3}{2}m_1R^2\ddot{\theta} = N_2\left(R+\varepsilon\cos\theta\right) - f_2\,\varepsilon\sin\theta$$

For $q_2 = x$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = Q_2 \qquad \frac{\partial L}{\partial\dot{x}} = m_2\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = m_2\ddot{x} \qquad \frac{\partial L}{\partial x} = 0$$

$$m_2\ddot{x} = F - N_2$$

We still need to determine f_2 – magnitude is $\mu|N_2|$ and direction opposes motion

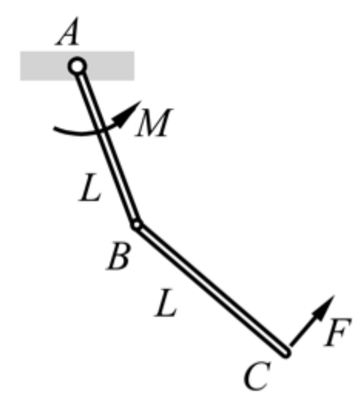
$$f_2 = \mu|N_2|\,\mathrm{sgn}\left(-\bar{v}_B\cdot\hat{j}\right) = \mu|N_2|\,\mathrm{sgn}\left(\varepsilon\dot{\theta}\sin\theta\right)$$

So, the full set of equations of motion is

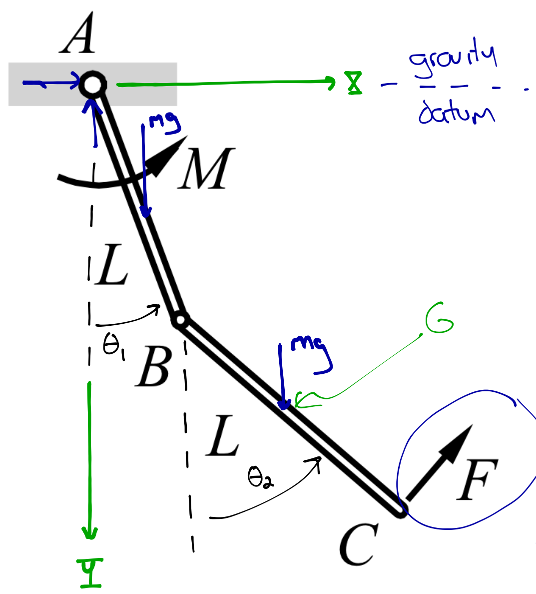
$$\boxed{\begin{aligned}
&\frac{3}{2}m_1R^2\ddot{\theta} = N_2\left(R+\varepsilon\cos\theta\right) - \mu|N_2|\,\mathrm{sgn}\left(\varepsilon\dot{\theta}\sin\theta\right)\varepsilon\sin\theta\\
&m_2\ddot{x} = F - N_2\\
&\dot{x} - \left(R+\varepsilon\cos\theta\right)\dot{\theta} = 0
\end{aligned}}$$

Exercise 8.5

EXERCISE 8.5 A known couple $M(t)$ is applied to the upper bar. Force F, which is applied perpendicularly to the lower bar, acts to make the velocity of end C always be parallel to the line from joint A to end B. The bars have equal mass m, and the system lies in the vertical plane. Use the method of Lagrange multipliers to derive the equations of motion.



Exercise 8.5



Exercise 8.5

This force enforces the constraint that $\bar{v}_C = v_C \bar{e}_{B/A}$

It's a constraint force.

Define $\bar{q} = (\theta_1, \theta_2)$

The constraint is $\bar{v}_C = v_C \bar{e}_{B/A}$ or $\bar{v}_C \times \bar{r}_{B/A} = 0$

Let's write $\bar{v}_C$

$\bar{v}_C = \bar{v}_B + -\dot{\theta}_2 \bar{K} \times \bar{r}_{C/B}$ and $\bar{v}_B = \cancel{\bar{v}_A}^{\,0} + (-\dot{\theta}_1 \bar{K}) \times \bar{r}_{B/A}$

$\bar{r}_{C/B} = L\sin\theta_2 \bar{I} + L\cos\theta_2 \bar{J}$ $\quad \bar{r}_{B/A} = L\sin\theta_1 \bar{I} + L\cos\theta_1 \bar{J}$

So $\bar{v}_B = -L\dot{\theta}_1 \sin\theta_1 \bar{J} + L\dot{\theta}_1 \cos\theta_1 \bar{I}$ and

$\bar{v}_C = (L\dot{\theta}_1\cos\theta_1 + L\dot{\theta}_2\cos\theta_2)\bar{I} + (-L\dot{\theta}_1\sin\theta_1 - L\dot{\theta}_2\sin\theta_2)\bar{J}$

$$\bar{v}_C \times \bar{r}_{B/A} = \left[(L\dot{\theta}_1\cos\theta_1 + L\dot{\theta}_2\cos\theta_2)\bar{I} + (-L\dot{\theta}_1\sin\theta_1 - L\dot{\theta}_2\sin\theta_2)\bar{J}\right] \times \left[L\sin\theta_1\bar{I} + L\cos\theta_1\bar{J}\right] = 0$$

$$= L\cos\theta_1(L\dot{\theta}_1\cos\theta_1 + L\dot{\theta}_2\cos\theta_2)\bar{K} + L\sin\theta_1(L\dot{\theta}_1\sin\theta_1 + L\dot{\theta}_2\sin\theta_2)\bar{K} = 0$$

$$= L^2\left[\dot{\theta}_1\cos^2\theta_1 + \dot{\theta}_2\cos\theta_1\cos\theta_2 + \dot{\theta}_1\sin^2\theta_1 + \dot{\theta}_2\sin\theta_1\sin\theta_2\right]\bar{K} = 0$$

$$= L^2\left[\dot{\theta}_1\underbrace{(\cos^2\theta_1 + \sin^2\theta_1)}_{=1} + \dot{\theta}_2\underbrace{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}_{=\cos(\theta_2-\theta_1)}\right]\bar{K} = 0$$

So, the velocity constraint is:

$\dot{\theta}_1 + \dot{\theta}_2\cos(\theta_2 - \theta_1) = 0 \longrightarrow$ match terms to $a_{11}\dot{\theta}_1 + a_{12}\dot{\theta}_2 + b_1 = 0$

$a_{11} = 1 \quad a_{12} = \cos(\theta_2 - \theta_1) \quad b_1 = 0$

Exercise 8.5 (cont.)

Now, we need to form the Lagrangian.

The upper bar is in pure rotation about A. For the lower, we need both linear and rotational components

$$T = \frac{1}{2} I_{22}^{1} \dot{\theta}_1^2 + \frac{1}{2} m \bar{v}_G \cdot \bar{v}_G + \frac{1}{2} I_{22}^{2} \dot{\theta}_2^2$$

$I_{22}^{1} \equiv$ moment of inertia of upper bar about A
$I_{22}^{2} \equiv$ moment of inertia of lower bar about G

$$\bar{v}_G = \left(L\dot{\theta}_1 \cos\theta_1 + \frac{L}{2}\dot{\theta}_2 \cos\theta_2 \right)\bar{I} + \left(-L\dot{\theta}_1 \sin\theta_1 - \frac{L}{2}\dot{\theta}_2 \sin\theta_2 \right)\bar{J}$$

← follows same form as $\bar{v}_C$ but $\bar{r}_{G/B} = \frac{1}{2}\bar{r}_{C/B}$
If you don't recognize that, solve for
$\bar{v}_G = \bar{v}_B + \bar{\omega} \times \bar{r}_{G/B}$

$$V = \underbrace{\left(-mg\frac{L}{2}\cos\theta_1 \right)}_{\text{upper bar}} + \underbrace{\left(-mg\left(L\cos\theta_1 + \frac{L}{2}\cos\theta_2 \right) \right)}_{\text{lower bar}}$$

$$L = T - V$$

Only M does virtual work. F is a constraint force, which we'll handle with Lagrange mult.

$$\delta W = M\delta\theta_1 \rightarrow Q_1 = M \text{ and } Q_2 = 0$$

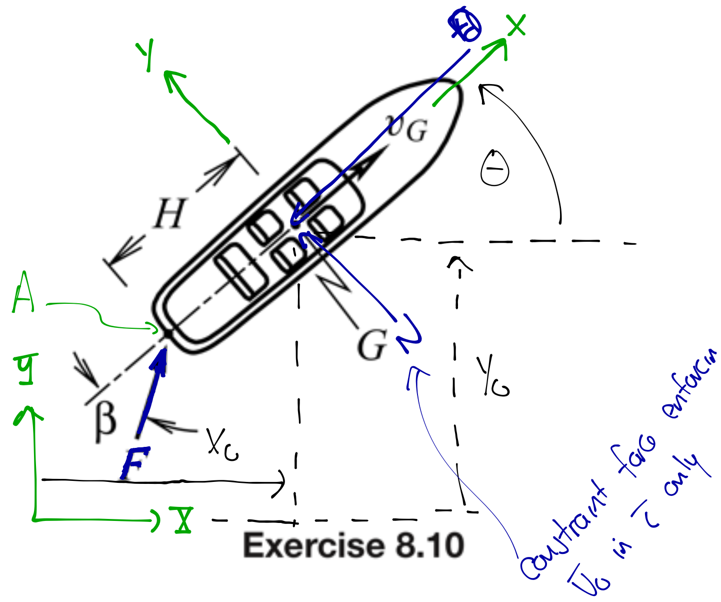
Now, solve Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = Q_1 + a_{11}\lambda_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = Q_2 + a_{12}\lambda_1$$

Exercise 8.10

EXERCISE 8.10 The thrust of an outboard motor on a boat may be represented as a force $\bar{F}$ acting at an angle β relative to the axis of the boat. The hydrodynamic properties of the boat are such that the velocity of the center of mass G is constrained to be parallel to the longitudinal axis of the boat. The component of the hydrodynamic force parallel to the axis of the boat is the drag f_d. Derive the equations of motion for the boat by using Lagrange multipliers. The mass of the boat is m, and its centroidal moment of inertia is I.



Exercise 8.10

Use $\bar{q} = (X_G, Y_G, \theta)$

Constraint is that $\bar{v}_G = \dot{X}_G \bar{I} + \dot{Y}_G \bar{J} = v_G \bar{i}$ ← or $\bar{v}_G \cdot \bar{j} = 0$

$$\bar{v}_G \cdot \bar{j} = \left(\dot{X}_G \bar{I} + \dot{Y}_G \bar{J}\right) \cdot \left(-\sin\theta \bar{I} + \cos\theta \bar{J}\right) = -\dot{X}_G \sin\theta + \dot{Y}_G \cos\theta = 0$$

← match to $a_{11}\dot{X}_G + a_{12}\dot{Y}_G + a_{13}\dot{\theta} + b_1 = 0$

$a_{11} = -\sin\theta$, $a_{12} = \cos\theta$, $a_{13} = 0$, $b_1 = 0$

Need to find the virtual work done by F and f_d. N is a constraint force, so it's handled by Lagrange mult.

$$\delta W = \bar{F} \cdot \delta \bar{r}_A + \left(-f_d \bar{i} \cdot \delta \bar{r}_G\right)$$

$$\bar{v}_A = \bar{v}_G + \bar{\omega} \times \bar{r}_{A/G} \qquad \bar{\omega} = \dot{\theta}\bar{k} = \dot{\theta}\bar{K} \qquad \bar{r}_{A/G} = -H\bar{i}$$

$$= \left[\dot{X}_G \bar{I} + \dot{Y}_G \bar{J}\right] + \left[\dot{\theta}\bar{k} \times -H\bar{i}\right] = \left[\dot{X}_G \bar{I} + \dot{Y}_G \bar{J}\right] + \left[-H\dot{\theta}\bar{j}\right]$$

$$= \left[\dot{X}_G \bar{I} + \dot{Y}_G \bar{J}\right] + \left[-H\dot{\theta}\left(-\sin\theta \bar{I} + \cos\theta \bar{J}\right)\right]$$

$$= \left(\dot{X}_G + H\dot{\theta}\sin\theta\right)\bar{I} + \left(\dot{Y}_G - H\dot{\theta}\cos\theta\right)\bar{J}$$

So,

$$\delta \bar{r}_A = \left(\delta X_G + H\sin\theta\, \delta\theta\right)\bar{I} + \left(\delta Y_G - H\cos\theta\, \delta\theta\right)\bar{J}$$

$$\bar{F} = F\cos(\theta + \beta)\bar{I} + F\sin(\theta + \beta)\bar{J}$$

$$\delta \bar{r}_G = \delta X_G \bar{I} + \delta Y_G \bar{J}$$

$$\delta W = \left(F\cos(\theta+\beta)\bar{I} + F\sin(\theta+\beta)\bar{J}\right) \cdot \left(\left(\delta X_G + H\sin\theta\, \delta\theta\right)\bar{I} + \left(\delta Y_G - H\cos\theta\, \delta\theta\right)\bar{J}\right)$$
$$+ \left(-f_d \cos\theta \bar{I} - f_d \sin\theta \bar{J}\right) \cdot \left(\delta X_G \bar{I} + \delta Y_G \bar{J}\right)$$

Exercise 8.10 (cont.)

$$\delta W = \left(F\cos(\theta+\beta)\bar{I} + F\sin(\theta+\beta)\bar{J}\right)\cdot\left(\left(\delta\bar{x}_G + H\sin\theta\,\delta\bar{\theta}\right)\bar{I} + \left(\delta\bar{y}_G - H\cos\theta\,\delta\theta\right)\bar{J}\right)$$
$$+ \left(-f_d\cos\theta\,\bar{I} - f_d\sin\theta\,\bar{J}\right)\cdot\left(\delta x_G\bar{I} + \delta y_G\bar{J}\right)$$

$$= F\cos(\theta+\beta)\,\delta\bar{x}_G + F\cos(\theta+\beta)H\sin\theta\,\delta\bar{\theta} + F\sin(\theta+\beta)\,\delta\bar{y}_G - FH\sin(\theta+\beta)\cos\theta\,\delta\theta$$
$$- f_d\cos\theta\,\delta\bar{x}_G - f_d\sin\theta\,\delta\bar{y}_G$$

$$\delta W = \underbrace{\left(F\cos(\theta+\beta) - f_d\cos\theta\right)}_{Q_1}\delta\bar{x}_G + \underbrace{\left(F\sin(\theta+\beta) - f_d\sin\theta\right)}_{Q_2}\delta\bar{y}_G + \underbrace{\left(FH\cos(\theta+\beta)\sin\theta - FH\sin(\theta+\beta)\cos\theta\right)}_{Q_3}\delta\theta$$

should simplify by trig identities

$$T = \tfrac{1}{2}m\bar{v}_G\cdot\bar{v}_G + \tfrac{1}{2}I_{zz}\dot{\theta}^2$$ $I_{zz} \equiv$ moment of inertia about G

$$= \tfrac{1}{2}m\left(\dot{x}_G\bar{I} + \dot{y}_G\bar{J}\right)\cdot\left(\dot{x}_G\bar{I} + \dot{y}_G\bar{J}\right) = \tfrac{1}{2}I_{zz}\dot{\theta}^2$$

$$= \tfrac{1}{2}m\left(\dot{x}_G^2 + \dot{y}_G^2\right) + \tfrac{1}{2}I_{zz}\dot{\theta}^2$$

$$V = 0$$

Now, form Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_G}\right) - \frac{\partial L}{\partial x_G} = Q_1 + a_{11}\lambda_1 \longrightarrow m\ddot{x}_G = F\cos(\theta+\beta) - f_d\cos\theta + (-\sin\theta)\lambda_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_G}\right) - \frac{\partial L}{\partial y_G} = Q_2 + a_{12}\lambda_1 \longrightarrow m\ddot{y}_G = F\sin(\theta+\beta) - f_d\sin\theta + (\cos\theta)\lambda_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_3 + a_{13}\lambda_1 \longrightarrow I_{zz}\ddot{\theta} = FH\cos(\theta+\beta)\sin\theta - FH\sin(\theta+\beta)\cos\theta$$