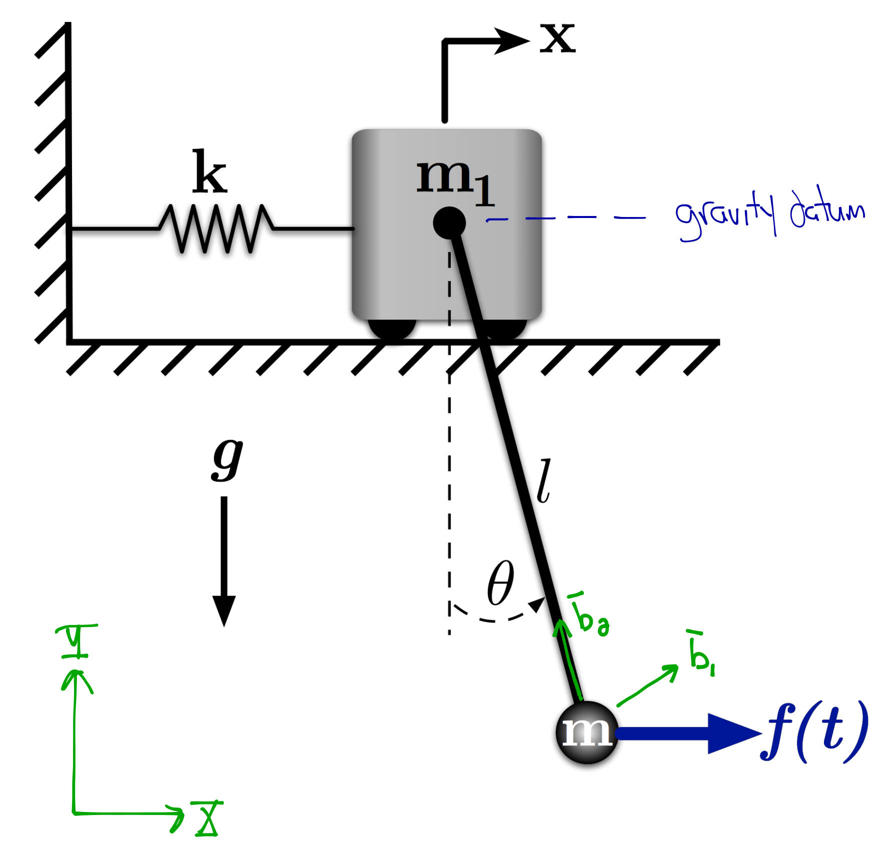
Example



2 bodies, 2 DOF → 2 generalized coords and 2 Eq of Motion

Use generalized coords (x, θ)

The velocity of m_1 is $\dot{x}$.

Q: What is the velocity of m?

It depends on both x and θ

$$\bar{V}_2 = \dot{x}\bar{I} + l\dot{\theta}\bar{b}_1 \quad \leftarrow \quad \bar{b}_1 = \cos\theta\,\bar{I} + \sin\theta\,\bar{J}$$

$$= (\dot{x} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}$$

Q: What should we choose as the gravity datum?

Let's choose the attachment point

(The book chooses the "lowest" pendulum point)

Now, form the energies, then the Lagrangian.

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m\bar{V}_2^T\bar{V}_2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right]\cdot\left[(\dot{x} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right]$$

$\left(\text{or } \frac{1}{2}m\bar{V}_2\cdot\bar{V}_2\right)$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right] \quad \leftarrow \quad \text{Total kinetic energy}$$

$$V = V_{gr} + V_{sp} \qquad V_{gr} = mgh = -mgl\cos\theta \qquad \text{and} \qquad V_{sp} = \frac{1}{2}kx^2$$

$$V = -mgl\cos\theta + \frac{1}{2}kx^2 \quad \leftarrow \quad \text{Total potential energy}$$

$$L = T - V \quad \leftarrow \quad \text{Lagrangian}$$

Q: What about the external force, $f(t)$? ← We have to find the Q_i terms for Lagrange's Eq.

To do so, we'll use virtual displacements and virtual work

We know that $\bar{V}_2 = (\dot{x} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}$ ← This is the point the force is applied. We'll use this velocity to figure out the virtual displacements.

$$\frac{d\bar{r}_2}{dt} = \left(\frac{dx}{dt} + l\frac{d\theta}{dt}\cos\theta\right)\bar{I} + \left(l\frac{d\theta}{dt}\sin\theta\right)\bar{J} \quad \leftarrow \quad \text{multiply by } dt \text{ and replace } dq_i \text{ with } \delta q_i$$

$$\delta\bar{r}_2 = (\delta x + l\,\delta\theta\cos\theta)\bar{I} + (l\,\delta\theta\sin\theta)\bar{J} \quad \leftarrow \quad \text{virtual displacement of } m$$

Example (cont.)

To find Q_i, we need to find the virtual work done by the force $(= \text{Force} \cdot \text{virtual displacement})$

$$\delta W = f\bar{I}\cdot\delta\bar{r}_2 = f\bar{I}\cdot\left[(\delta x + l\delta\theta\cos\theta)\bar{I} + (l\delta\theta\sin\theta)\bar{J}\right] = F(\delta x + l\delta\theta\cos\theta)$$

$$\delta W = \underbrace{F}_{Q_1}\underbrace{\delta x}_{\delta q_1} + \underbrace{(Fl\cos\theta)}_{Q_2}\underbrace{\delta\theta}_{\delta q_2}$$

Now, apply Lagrange's Eq for each q_i - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$

$q_1 = x$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_1$$

$$\frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + \frac{1}{2}m\left[2(\dot{x} + l\dot{\theta}\cos\theta)\right] \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1\ddot{x} + m(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) \qquad Q_1 = f$$

$$\boxed{(m_1+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml^2\dot{\theta}^2\sin\theta + kx = F}$$

$q_2 = \theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_2$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left[2(\dot{x} + l\dot{\theta}\cos\theta)(l\cos\theta) + 2(l\dot{\theta}\sin\theta)(l\sin\theta)\right]$$

$$= m\left[l\dot{x}\cos\theta + l^2\dot{\theta}\cos^2\theta + l^2\dot{\theta}\sin^2\theta\right] = m(l\dot{x}\cos\theta + l^2\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m(l\ddot{x}\cos\theta - l\dot{x}\dot{\theta}\sin\theta + l^2\ddot{\theta})$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}m\left[2(\dot{x} + l\dot{\theta}\cos\theta)(-l\dot{\theta}\sin\theta) + 2(l\dot{\theta}\sin\theta)(l\dot{\theta}\cos\theta)\right] - mgl\sin\theta$$

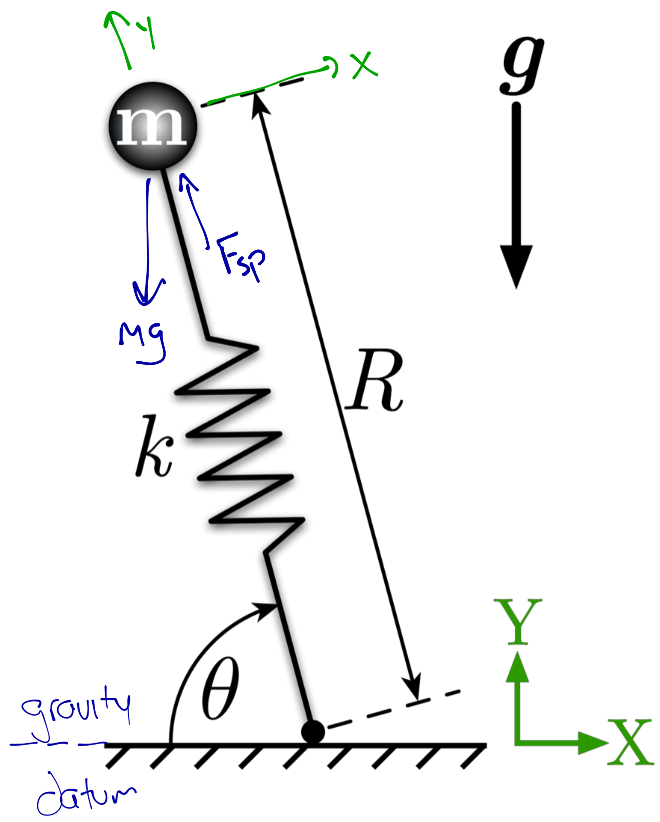
$$= m\left[-l\dot{x}\dot{\theta}\sin\theta - gl\sin\theta\right]$$

$$Q_2 = fl\cos\theta$$

$$m(l\ddot{x}\cos\theta - \cancel{l\dot{x}\dot{\theta}\sin\theta} + l^2\ddot{\theta}) - m(\cancel{-l\dot{x}\dot{\theta}\sin\theta} - gl\sin\theta) = fl\cos\theta$$

$$\boxed{ml^2\ddot{\theta} + ml\ddot{x}\cos\theta + mgl\sin\theta = fl\cos\theta}$$

Example



Spring Loaded Inverted Pendulum (SLIP) Model often used to model walking

Here, we'll assume we're always in contact with the ground

This would be called the stance phase of locomotion

$L_0 \equiv$ equil length of the spring

$\bar{q} = (R, \theta)$

$\bar{v}_M = R\dot{\theta}\,\hat{i} + \dot{R}\,\hat{j}$

$T = \frac{1}{2}m\,\bar{v}_M \cdot \bar{v}_M = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{R}^2\right)$

$V = U_{grav} + U_{sp}$

$U_{grav} = mgR\sin\theta \qquad U_{sp} = \frac{1}{2}k\Delta^2 = \frac{1}{2}k(R-L_0)^2$

$V = mgR\sin\theta + \frac{1}{2}k(R-L_0)^2$

$L = T - V = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{R}^2 - mgR\sin\theta - \frac{1}{2}k(R-L_0)^2$

Now, write Lagrange's Equations for each gen. coord

for $q_1 = R$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = 0$$

$\frac{\partial L}{\partial \dot{R}} = m\dot{R} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = m\ddot{R} \qquad \frac{\partial L}{\partial R} = 2mR\dot{\theta}^2 - mg\sin\theta + k(R-L_0)$

$$\boxed{m\ddot{R} - 2mR\dot{\theta}^2 + mg\sin\theta - k(R-L_0) = 0}$$

Example (cont.)

$$L = T - V = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{R}^2 - mgR\sin\theta - \frac{1}{2}k(R-L_0)^2$$

for $q_2 = \theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

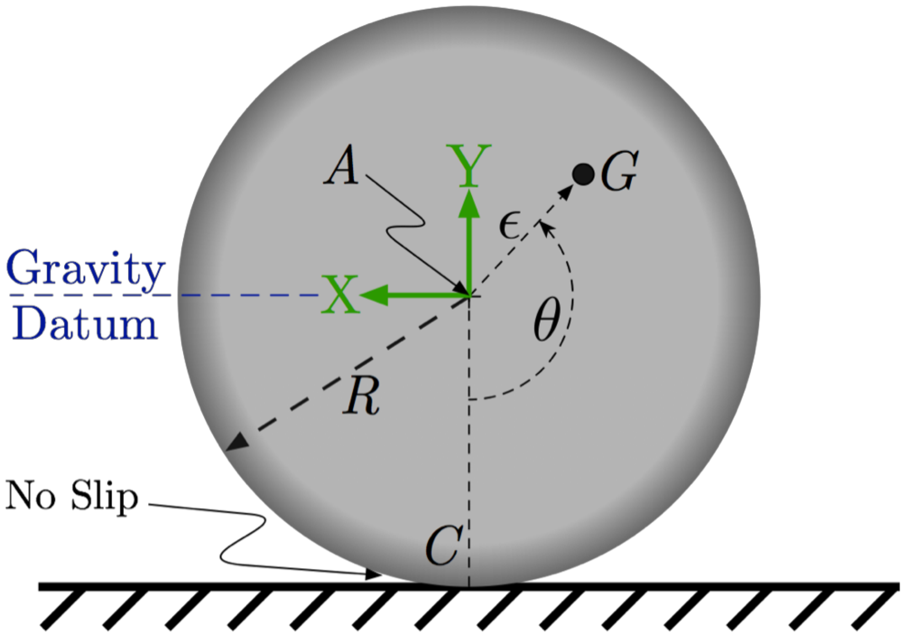
$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgR\cos\theta$$

$$\boxed{2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta} + mgR\cos\theta = 0}$$

Q: How would we handle integrating the flight phase? (when foot/spring leaves the ground)

Generate a separate model for that phase. In simulation, we'd need to check for the spring going into extension at each step. At that transition, we'd switch to the flight model, using the system states at that transition as the initial conditions for the flight phase of simulation. During the flight phase of the simulation, we'd check for ground contact at each step of the solution. Once we see contact, we'd use the state at contact as the initial condition for the next stance phase. This process would repeat over the simulation duration.

Example



Want the equations of motion

$\bar{V}_A = R\dot{\theta}\bar{I}$

$\bar{V}_G = \bar{V}_A + \bar{\omega}\times\bar{r}_{G/A}$ $\qquad \bar{\omega} = -\dot{\theta}\bar{K}$

$= R\dot{\theta}\bar{I} + \left(-\dot{\theta}\bar{K}\times\epsilon\left(-\sin\theta\bar{I} - \cos\theta\bar{J}\right)\right)$

$= R\dot{\theta}\bar{I} + \left(\epsilon\dot{\theta}\sin\theta\bar{J} - \epsilon\dot{\theta}\cos\theta\bar{I}\right)$

$\bar{V}_G = \left(R\dot{\theta} - \epsilon\dot{\theta}\cos\theta\right)\bar{I} + \left(\epsilon\dot{\theta}\sin\theta\right)\bar{J}$

For kinetic energy, we need to consider both translation and rotation

$T = \frac{1}{2}I_{disk_{zz}}\dot{\theta}^2 + \frac{1}{2}m\bar{V}_G\cdot\bar{V}_G = \frac{1}{2}I_{zz}\dot{\theta}^2 + \frac{1}{2}m\left(\left(R\dot{\theta} - \epsilon\dot{\theta}\cos\theta\right)^2 + \left(\epsilon\dot{\theta}\sin\theta\right)^2\right)$

$= \frac{1}{2}I_{zz}\dot{\theta}^2 + \frac{1}{2}m\left(R^2\dot{\theta}^2 - 2R\epsilon\dot{\theta}^2\cos\theta + \epsilon^2\dot{\theta}^2\cos^2\theta + \epsilon^2\dot{\theta}^2\sin^2\theta\right)$

$= \frac{1}{2}I_{zz}\dot{\theta}^2 + \frac{1}{2}m\left(R^2\dot{\theta}^2 + \epsilon^2\dot{\theta}^2 - 2R\epsilon\dot{\theta}^2\cos\theta\right)$

$T = \frac{1}{2}\left(I_{zz} + mR^2 + m\epsilon^2\right)\dot{\theta}^2 - R\epsilon\dot{\theta}^2\cos\theta$

Aside: $Q_1 = 0$... why?

If assuming no slip, then neither f or N do virtual work.

Only potential energy is from the eccentricity,

$V = mgh = mg\left(-\epsilon\cos\theta\right)$

So, $L = T - V$

$L = \frac{1}{2}\left(I_{zz} + mR^2 + m\epsilon^2\right)\dot{\theta}^2 - mR\epsilon\dot{\theta}^2\cos\theta + mg\epsilon\cos\theta$

for $q = \theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$\frac{\partial L}{\partial\dot{\theta}} = \left(I_{zz} + mR^2 + m\epsilon^2\right)\dot{\theta} - 2mR\epsilon\dot{\theta}\cos\theta$ $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \left(I_{zz} + mR^2 + m\epsilon^2\right)\ddot{\theta} - 2mR\epsilon\ddot{\theta}\cos\theta + 2mR\epsilon\dot{\theta}^2\sin\theta$

$\frac{\partial L}{\partial\theta} = mR\epsilon\dot{\theta}^2\sin\theta - mg\epsilon\sin\theta$

$$\boxed{\left(I_{zz} + mR^2 + m\epsilon^2 - 2mR\epsilon\cos\theta\right)\ddot{\theta} + mR\epsilon\dot{\theta}^2\sin\theta + mg\epsilon\sin\theta = 0}$$