Relation Between Constraint Forces and Conditions (Sec. 7.4.2)

A force or couple is a constraint force associated with a kinematical constraint condition if, and only if, it does no work in a virtual movement that is consistent with the constraint.

(All from page 432 of your book)

The virtual work done by constraint forces always is zero when unconstrained generalized coordinates are used to describe a holonomic system.

When constrained generalized coordinates are used to describe the position of a system, constraint forces will do virtual work.

Define $R_j^{(i)}$, $i = 1, \dots J$ ← contribution of i^{th} constraint to j^{th} gen. force

In config space, generalized forces can be written as vector $\hat{Q}$, ← hat indicates config space

Define

$$\hat{R}^i = R_1^i \hat{e}_1 + \dots + R_N^i \hat{e}_n$$ ← portion of $\hat{Q}$ from constraint

Select virtual displacement that is consistent with constraint, $\delta\hat{r}$, then it must be $\perp$ to normal direction $\hat{a}_i$ associated Jacobian constraint coeff.

So, $\hat{a}_i \cdot \delta\hat{r} = 0$

If $\delta\hat{r}$ is consistent with constraint, then constraint force does no virtual work

So, $\hat{R}^i \cdot \delta\hat{r} = 0$

If $\hat{a}_i \cdot \delta\hat{r} = 0$, then $\hat{R}^i \cdot \delta\hat{r} = 0$ ← True for any $\delta\hat{r} \perp \hat{a}_i$ → ∞ many possibilite

$\hat{R}^i = \hat{a}_i \lambda_i$ ← So, $\hat{R}^i \parallel \hat{a}_i$

$\lambda_i \equiv$ Lagrange Multiplier

Sub back into full gen. coord form

$R_j^{(i)} = a_{ij} \lambda_i$ ← We can describe contributions from constraint forces by knowing Jacobian constraint matrix

Usually easier than evaluating their virtual work

Relation Between Constraint Forces and Conditions (cont.)

What if we want to know the actual constraint force, rather than use multiplier...

Need to evaluate the virtual work done by the force.

$\bar{e}_i \equiv$ direction in which motion is limited by the constraint ← Also the direction the constraint force acts

So, i^{th} constraint force/moment is $C_i \bar{e}_i$ where $|C_i|$ is unknown

The virtual work done by it is

$$\delta W_i = C_i \sum_{j=1}^{N} c_{ij} \delta q_j$$

← just saying that it's a linear in δq_j

c_{ij} is function of gen coord and time

Look at this contribution to gen forces Q_j

$$\delta W = \sum_{j=1}^{N} Q_j \delta q_j = C_i \sum_{j=1}^{N} c_{ij} \delta q_j$$

Then

$$R_j^{(i)} = c_{ij} C_i$$

Compare to Lagrange Multiplier

$$R_j^{(i)} = c_{ij} C_i = a_{ij} \lambda_i \longrightarrow C_i = \frac{a_{ij}}{c_{ij}} \lambda_i$$

This must hold for any gen coord j

$$C_i = \sigma_i \lambda_i \quad \sigma_i = \frac{a_{ij}}{c_{ij}}$$

To find σ_i, we need to find c_{ij}

Relation Between Constraint Forces and Conditions (cont.)

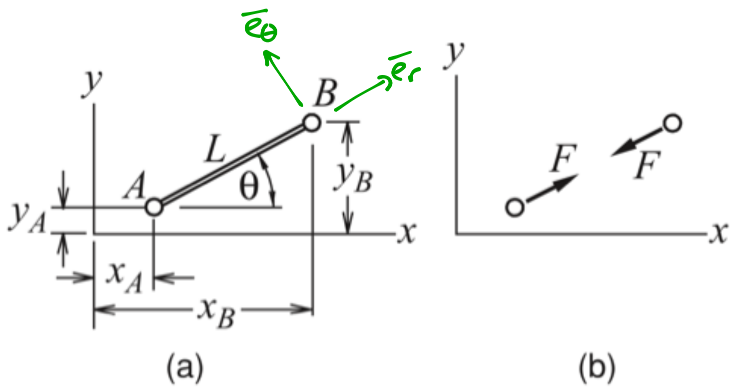

Figure 7.10. Generalized coordinates for a system consisting of two spheres connected by a rigid bar.

Case 1

Define $\bar{q} = (x_A, y_A, \theta)$ ← unconstrained set of gen. coord. ← Constraint forces should do no virtual work

$$\bar{v}_B = \bar{v}_A + \dot{\theta}\bar{k} \times \bar{r}_{B/A} = v_A + L\dot{\theta}\bar{e}_\theta \longrightarrow \delta\bar{r}_B = \delta\bar{r}_A + L\delta\theta\,\bar{e}_\theta$$

So the virtual work by F is then

$$\delta W = \left(-F\bar{e}_r \cdot \delta\bar{r}_A\right) + \left(F\bar{e}_r \cdot \delta\bar{r}_B\right)$$
$$= \left(-F\bar{e}_r \cdot \delta\bar{r}_A\right) + \left(F\bar{e}_r \cdot \left(\delta\bar{r}_A + L\delta\theta\,\bar{e}_\theta\right)\right)$$
$$= F\bar{e}_r \cdot L\delta\theta\,\bar{e}_\theta = 0$$

← F does no work because it is the constraint force maintaining L = constant

Case 2

Define $\bar{q} = (x_A, y_A, x_B, y_B)$ ← constrained set of gen. coord ← Constraint forces will do virtual work

Q: How can we write the constraint that L = const in terms of these gen coord?

$$(x_B - x_A)^2 + (y_B - y_A)^2 = L^2$$

write the velocity form of this equation

$$\cancel{2}(x_B - x_A)(\dot{x}_B - \dot{x}_A) + \cancel{2}(y_B - y_A)(\dot{y}_B - \dot{y}_A) = 0 \iff a_{11}\dot{x}_A + a_{12}\dot{y}_A + a_{13}\dot{x}_B + a_{14}\dot{y}_B = 0$$

so

$$a_{11} = -a_{13} = x_A - x_B \quad \text{and} \quad a_{12} = -a_{14} = y_A - y_B$$

Case 2 (cont.)

Since we are using the constrained gen coord., we write the virtual displacements independently

$$\delta \bar{r}_A = \delta x_A \bar{i} + \delta y_A \bar{j} \qquad \text{and} \qquad \delta \bar{r}_B = \delta x_B \bar{i} + \delta y_B \bar{j}$$

So, the virtual work done by F can be written as

$$\delta W = \left(-F\bar{e}_r \cdot \delta \bar{r}_A\right) + \left(F\bar{e}_r \cdot \delta \bar{r}_B\right) = F\cos\theta\left(\delta x_B - \delta x_A\right) + F\sin\theta\left(\delta y_B - \delta y_A\right)$$

θ is not a generalized coord, so we must eliminate it

Can see from geometry that

$$\cos\theta = \frac{x_B - x_A}{L}$$

$$\sin\theta = \frac{y_B - y_A}{L}$$

$$\delta W = F\left(\frac{x_B - x_A}{L}\right)\left(\delta x_B - \delta x_A\right) + F\left(\frac{y_B - y_A}{L}\right)\left(\delta y_B - \delta y_A\right)$$

Now, match terms to

$$\delta W = R_1^{(1)} \delta x_A + R_2^{(1)} \delta y_A + R_3^{(1)} \delta x_B + R_4^{(1)} \delta y_B \quad \text{to find that}$$

$$R_1^{(1)} = -R_3^{(1)} = -\frac{x_B - x_A}{L} F \qquad \text{and} \qquad R_2^{(1)} = -R_4^{(1)} = -\frac{y_B - y_A}{L} F$$

Virtual work is nonzero, as expected... Also match earlier (general) analysis

$$R_j^{(i)} = c_{ij} C_i$$

$C = F \rightarrow$

$$C_{11} = C_{13} = -\frac{x_B - x_A}{L} \qquad \text{and} \qquad C_{12} = C_{14} = -\frac{y_B - y_A}{L}$$

and $\dfrac{a_{0j}}{c_{j1}} = \sigma_1 = L$

Chapter 8 - Constrained Generalized Coordinates

Lagrange's Equations - Constrained Case (Sec. 8.1)

Choice of whether to use constrained sets of gen. coords or not often comes down to the tradeoff between ease of forming the equations and the ease of solving them

We've seen that we can write velocity constraints (or config constraints in velocity form) as

$$\sum_{j=1}^{N} a_{ij}\dot{q}_j + b_i = 0 \qquad i = 1, \dots, J$$

$J \equiv$ number of constraints
$N \equiv$ number of gen. coords
$a_{ij} \equiv$ Jacobian constraint coeff

We also wrote generalized forces (for nonconserv. forces) as

$Q_j \rightarrow$ split into applied Q_j^a and constraint R_j

$$Q_j = Q_j^a + R_j$$

Just scalar sum of contribution of each constraint for $R_j^{(i)}$

We saw that a_{ij} is enough to characterize this via a Lagrange Mult.

$$R_j^{(i)} = a_{ij}\lambda_i$$

So, we can use this to write Lagrange's Equation as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^a + \sum_{i=1}^{J} a_{ij}\lambda_i \qquad j = 1, \dots, N$$

Example of needing to include constraint force

For a system with Coloumb friction, we need to know the Normal force to calc. friction force

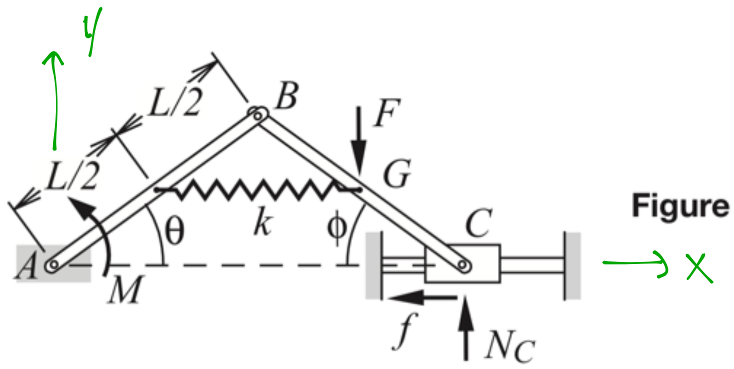

Figure 8.1. A linkage in which there is friction at collar *C*.

We need to use constrained coords... choose $\bar{q} = (\theta, \phi)$

Constraint is then $\theta - \phi = 0$

$$\bar{r}_{C/A} = L(\cos\theta + \cos\phi)\,\bar{i} + L(\sin\theta - \sin\phi)\,\bar{j}$$

$$\delta\bar{r}_C = \frac{\partial \bar{r}_{C/A}}{\partial \theta}\,\delta\theta + \frac{\partial \bar{r}_{C/A}}{\partial \phi}\,\delta\phi = L(-\sin\theta\,\bar{i} + \cos\theta\,\bar{j})\,\delta\theta + L(-\sin\phi\,\bar{i} - \cos\phi\,\bar{j})\,\delta\phi$$

So, the virtual work from the Normal force N_c and friction at C is then

$$\delta W = \left[-\mu_k |N_c|\, \mathrm{sgn}(\bar{v}_c \cdot \bar{i})\,\bar{i} + N_c\,\bar{j}\right] \cdot \delta\bar{r}_c$$

find that

$$Q_1 = \mu_k |N_c| L \sin\theta\, \mathrm{sgn}(\bar{v}_c \cdot \bar{i}) + N_c L \cos\theta$$

$$Q_2 = \mu_k |N_c| L \sin\phi\, \mathrm{sgn}(\bar{v}_c \cdot \bar{i}) - N_c L \cos\phi$$