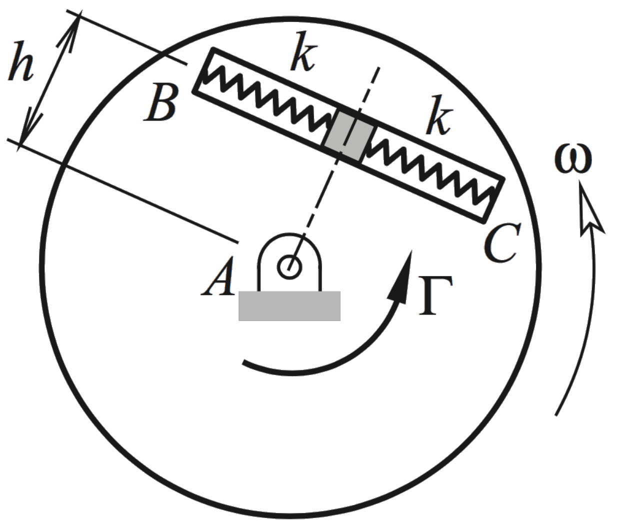
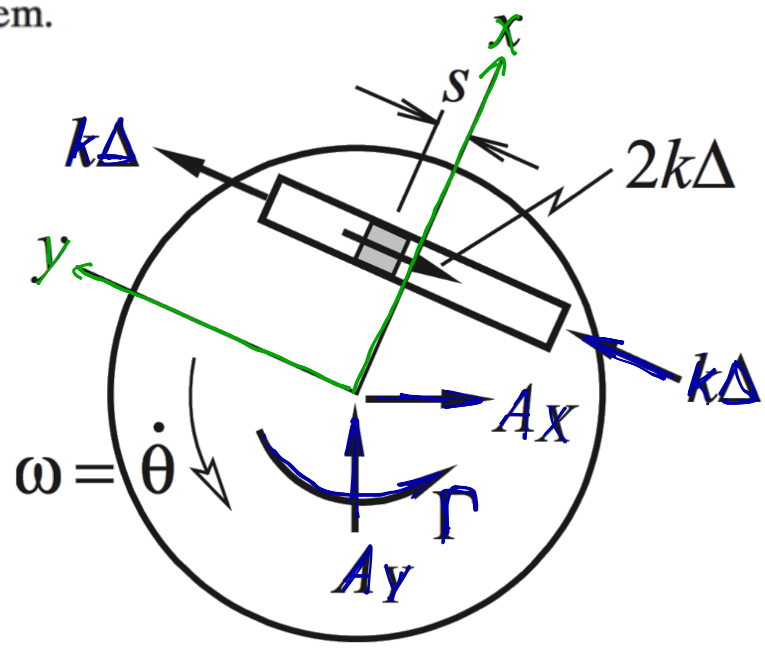
# Example 7.13

**EXAMPLE 7.13** The table rotates in a horizontal plane about bearing $A$ due to a torque $\Gamma$ whose time dependence is known. The mass of the table is $m_1$, and its radius of gyration about its center is $\kappa$. The slider, whose mass is $m_2$, moves within groove $BC$ under the restraint of two springs that are unstretched in the position shown. Derive the equations of motion for this system.




Q: What should we choose as generalized coordinates?

$\bar{q} = (\theta, s)$ s ≡ displacement of mass from eq. length of springs

Q: What generalized forces are there?

$\Gamma$ is only nonconservative force. It acts directly on $\theta$ so $\delta W = \Gamma \delta\theta \longrightarrow Q_1 = \Gamma \quad Q_2 = 0$

Q: What is the kinetic energy?

$$T = \underbrace{\frac{1}{2}(I_{zz})_{table}\,\omega^2}_{\text{Table in pure rotation}} + \underbrace{\frac{1}{2} m \bar{v}_s \cdot \bar{v}_s}_{\text{point mass total vel term}}$$

Q: What is $(I_{zz})_{table}$?

$(I_{zz})_{table} = Mk^2$ k ≡ radius of gyration given in problem

Q: What is $v_s$?

$$\bar{v}_s = \cancelto{0}{\bar{v}_A} + (\bar{v}_s)_{xyz} + \bar{\omega} \times \bar{r}_{s/A} \qquad \bar{r}_{s/A} = l\bar{i} + s\bar{j} \qquad (\bar{v}_s)_{xyz} = \dot{s}\bar{j} \qquad \bar{\omega} = \dot{\theta}\bar{k}$$

$$= \dot{s}\bar{j} + \left(\dot{\theta}k \times (l\bar{i} + s\bar{j})\right) = -s\dot{\theta}\bar{i} + (\dot{s} + l\dot{\theta})\bar{j}$$

# Example 7.13 (cont.)

So

$$T = \frac{1}{2} M k^2 \dot{\theta}^2 + \frac{1}{2} m \left[ (s\dot{\theta})^2 + (\dot{s} + l\dot{\theta})^2 \right]$$

Q: What is the potential energy?

$$V = \frac{1}{2} k \Delta_1^2 + \frac{1}{2} k \Delta_2^2 \qquad \Delta_1 = \Delta_2 = \Delta \longrightarrow V = k\Delta^2$$

We defined $s$ along the spring axis and as the deflection from spring eq., so

$$V = ks^2$$

$$L = \frac{1}{2} M k^2 \dot{\theta}^2 + \frac{1}{2} m \left[ (s\dot{\theta})^2 + (\dot{s} + l\dot{\theta})^2 \right] - ks^2$$

Now, apply Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad j = 1, \ldots, M \qquad \longleftarrow \text{Here } M = 2 \text{ and } \bar{q} = (\theta, s) \Big\} \text{ 2 equations}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_1 \quad \underline{\text{and}} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = Q_2$$

For $q_1 = \theta$

$$\frac{\partial L}{\partial \dot{\theta}} = M k^2 \dot{\theta} + \frac{1}{2} m \left[ 2 s^2 \dot{\theta} + 2(\dot{s} + l\dot{\theta}) l \right] = M k^2 \dot{\theta} + m s^2 \dot{\theta} + m l \dot{s} + m l^2 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = M k^2 \ddot{\theta} + m s^2 \ddot{\theta} + 2 m s \dot{s} \dot{\theta} + m l \ddot{s} + m l^2 \ddot{\theta} = (M k^2 + m s^2 + m l^2) \ddot{\theta} + 2 m s \dot{s} \dot{\theta} + m l \ddot{s}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\boxed{(M k^2 + m s^2 + m l^2) \ddot{\theta} + 2 m s \dot{s} \dot{\theta} + m l \ddot{s} = \Gamma}$$

For $q_2 = s$

$$\frac{\partial L}{\partial \dot{s}} = m \dot{s} + m l \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m \ddot{s} + m l \ddot{\theta} \qquad \boxed{m \ddot{s} + m l \ddot{\theta} - m s \dot{\theta}^2 + 2ks = 0}$$

$$\frac{\partial L}{\partial s} = m s \dot{\theta}^2 - 2ks$$

# Example

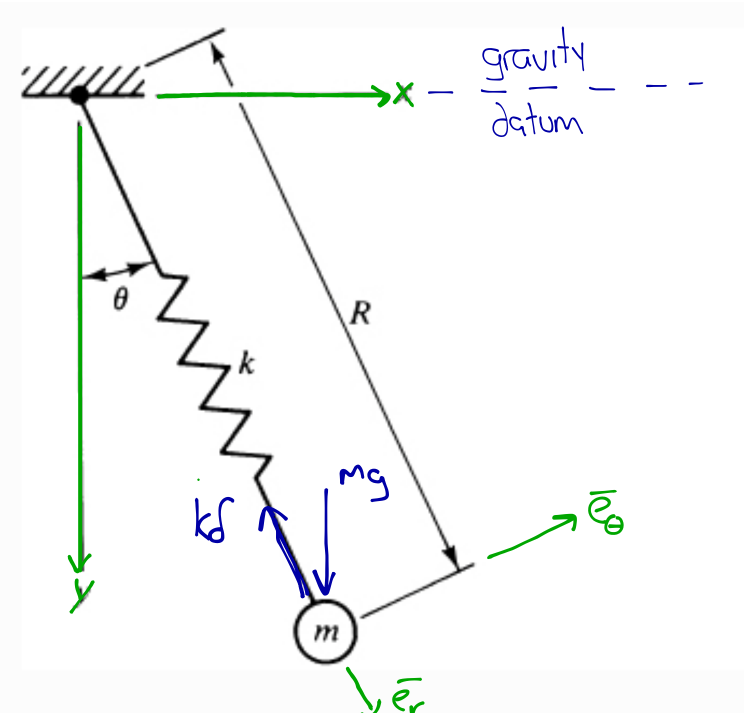



**Figure 7.3** Elastically supported pendulum.

Choose $q = (R, \theta)$ Unstretched spring Length $L_0$

$$T = \frac{1}{2}m\left(\dot{R}\bar{e}_r + R\dot{\theta}\bar{e}_\theta\right)\cdot\left(\dot{R}\bar{e}_r + R\dot{\theta}\bar{e}_\theta\right)$$

$$= \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2\right)$$

$$V = -mgR(\cos\theta) + \frac{1}{2}k(R-L_0)^2$$

$$L = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2\right) + mgR\cos\theta - \frac{1}{2}k(R-L_0)^2$$

for $q_1 = R$

$$\frac{\partial L}{\partial \dot{R}} = m\dot{R} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = m\ddot{R}$$

$$\frac{\partial L}{\partial R} = mR\dot{\theta}^2 + mg\cos\theta - k(R-L_0)$$

$$\boxed{m\ddot{R} - mR\dot{\theta}^2 - mg\cos\theta + k(R-L_0) = 0}$$

for $q_2 = \theta$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgR\sin\theta$$

$$\boxed{2m\dot{R}\dot{\theta} + mR\ddot{\theta} + mg\sin\theta = 0}$$

Q: How can we simulate this system?

ODE solvers usually want 1st-order ODEs

Define $x = \begin{bmatrix} R \\ \dot{R} \\ \theta \\ \dot{\theta} \end{bmatrix}$ and write $\dot{x} = f(x,t)$

$$\dot{x}_1 = \dot{R} = x_2$$

$$\dot{x}_2 = \ddot{R} = \frac{1}{m}\left[mR\dot{\theta}^2 + mg\cos\theta - k(R-L_0)\right] = \frac{1}{m}\left[mx_1x_4^2 + mg\cos x_3 - k(x_1-L_0)\right]$$

$$\dot{x}_3 = \dot{\theta} = x_4$$

$$\dot{x}_4 = \ddot{\theta} = \frac{1}{mR}\left[-2m\dot{R}\dot{\theta} - mgR\sin\theta\right]$$

# Linearization

*Lots* of controls and vibration analysis method depend on the sytsem being linear. No "real" system is totally linear, but we systems are often "near" linear in certain operating conditions and/or over small operating regimes. We can linearize a system about these points.

In many cases, it makes sense to linearize about an equilibrium position.

Q: How do we know what the equilibrium positions are?

One "trick" is to eliminate the "motion" variables (velocity and higher order derivatives) from the equations of motion.

$$\cancel{m\ddot{R}} - \cancel{mR\dot{\theta}^2} - mg\cos\theta + k(R-L_0) = 0 \longrightarrow R = L_0 + \frac{mg}{k}\cos\theta$$

$$\cancel{2m\dot{R}\dot{\theta}} + \cancel{mR\ddot{\theta}} + mg\sin\theta = 0 \longrightarrow \sin\theta = 0 \longrightarrow \theta = n\pi \quad n = 0, \dots$$

Q: For this system, which $\theta$ equil makes sense?

$\theta = 0$

So, the equil condition is $R_{eq} = L_0 + \frac{mg}{k}$ and $\theta_{eq} = 0$

Now, write the gen. coords in terms of the equil conditions

$$R = R_{eq} + \xi_1 \quad \text{and} \quad \theta = 0 + \xi_2$$

$$\dot{R} = \dot{\xi}_1, \quad \ddot{R} = \ddot{\xi}_1 \qquad \dot{\theta} = \dot{\xi}_2 \quad \ddot{\theta} = \ddot{\xi}_2$$

And, sub into the equations of motion:

$$m\ddot{\xi}_1 - mR\dot{\xi}_2^2 - mg\cos\xi_2 + k(R_{eq} + \xi_1 - L_0) = 0 \longrightarrow m\ddot{\xi}_1 - mR\dot{\xi}_2^2 - mg\cos\xi_2 + k\left(\xi_1 + \frac{mg}{k}\right) = 0$$

$$2m\dot{\xi}_1\dot{\xi}_2 + mR_{eq}\ddot{\xi}_2 + mg\sin\xi_2 = 0$$

← We are operating about the eq. so use $R_{eq}$ here

$\xi_1$ and $\xi_2 \ll 1$ so $\cos\xi_2 = 1 - \frac{\xi_2^2}{2} + \dots = 1$ and $\sin\xi_2 = \xi_2 + \frac{1}{3!}\xi_2^3 = \xi_2$

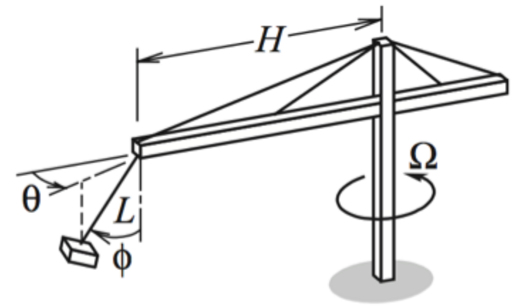
$\dot{\xi}_2^2 \ll \xi_2$ and $\dot{\xi}_1\dot{\xi}_2 \ll 1$ ← so ignore these terms

$$m\ddot{\xi}_1 - mg + k\xi_1 + mg = 0 \longrightarrow \boxed{m\ddot{\xi}_1 + k\xi_1 = 0}$$

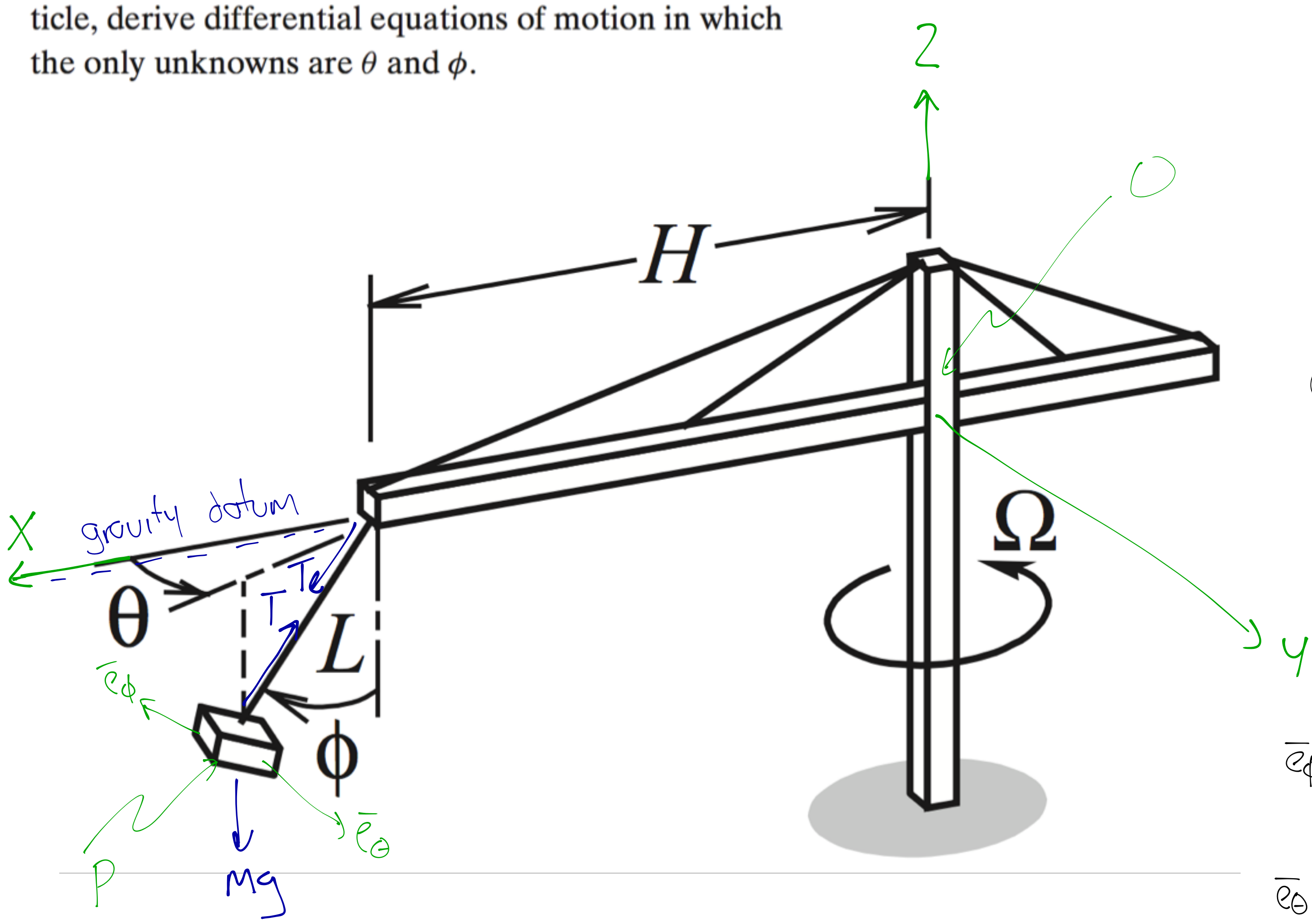
$$mR_{eq}\ddot{\xi}_2 + mg\xi_2 = 0 \longrightarrow \boxed{R_{eq}\ddot{\xi}_2 + g\xi_2 = 0}$$

## Problem 7.60

**EXERCISE 7.60** A shipping container is suspended from a crane by an inextensible cable. The crane rotates in the vertical plane at angular speed $\Omega$ whose time dependence is known. It may be assumed that the cable remains taut, so its orientation is describable in terms of the angle $\theta$ locating the vertical plane in which it is situated relative to the plane of the crane, and the angle of elevation $\phi$ from a vertical line. Based on a model of the container as a small particle, derive differential equations of motion in which the only unknowns are $\theta$ and $\phi$.




**Exercise 7.60**




xyz - fixed to jib (rotates with $\Omega$)

Choose:

$\bar{q} = (\theta, \phi)$

$\bar{e}_\phi = \sin\phi\,\bar{k} + \cos\phi(\cos\theta\,\bar{i} + \sin\theta)\bar{j}$

$\bar{e}_\theta = -\sin\theta\,\bar{i} + \cos\theta\,\bar{j}$

$T = \frac{1}{2} m \bar{v}_P \cdot \bar{v}_P$ and $U = mgh = -mgL\cos\phi$

$\bar{v}_P = \cancel{\bar{v}_O}^{0} + (v_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O}$

$(v_P)_{xyz}$ is most easily described in spherical coordinates

$$(v_P)_{xyz} = L\dot{\phi}\,\bar{e}_\phi + (L\sin\phi)\dot{\theta}\,\bar{e}_\theta = L\dot{\phi}\left(\sin\phi\,\bar{k} + \cos\phi\cos\theta\,\bar{i} + \cos\phi\sin\theta\,\bar{j}\right) + (L\sin\phi\,\dot{\theta})\left(-\sin\theta\,\bar{i} + \cos\theta\,\bar{j}\right)$$

$$\bar{r}_{P/O} = (H + L\sin\phi\cos\theta)\,\bar{i} + (L\sin\phi\sin\theta)\,\bar{j} + (-L\cos\phi\,\bar{k})$$

$$\bar{\omega} = \Omega\,\bar{k}$$

# Problem 7.60 (cont.)

$$\bar{v}_p = L\dot{\phi}\left(\sin\phi\,\bar{k} + \cos\phi\cos\theta\,\bar{i} + \cos\phi\sin\theta\,\bar{j}\right) + \left(L\sin\phi\,\dot{\theta}\left(-\sin\theta\,\bar{i} + \cos\theta\,\bar{j}\right)\right) +$$
$$\Omega\bar{k} \times \left(H + L\sin\phi\cos\theta\right)\bar{i} + \left(L\sin\phi\sin\theta\right)\bar{j} + \left(-L\cos\phi\,\bar{k}\right)$$

$$= \left[L\dot{\phi}\cos\phi\cos\theta\,\bar{i} - L\dot{\theta}\sin\phi\sin\theta - L\Omega\sin\phi\sin\theta\right]\bar{i}$$
$$+ \left[L\dot{\phi}\cos\phi\sin\theta + L\dot{\theta}\sin\phi\cos\theta + \Omega\left(H + L\sin\phi\cos\theta\right)\right]\bar{j}$$
$$+ \left[L\dot{\phi}\sin\phi\right]\bar{k}$$

$$T = \frac{1}{2}M\,\bar{v}_p \cdot v_p$$
$$L = T - V$$

Q: Are there any nonconservative forces?

No $\rightarrow$ $Q_i = 0 \quad \forall i$

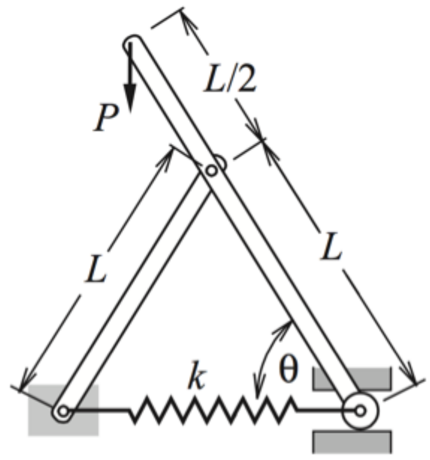
So,

for $q_1 = \theta \rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = 0$

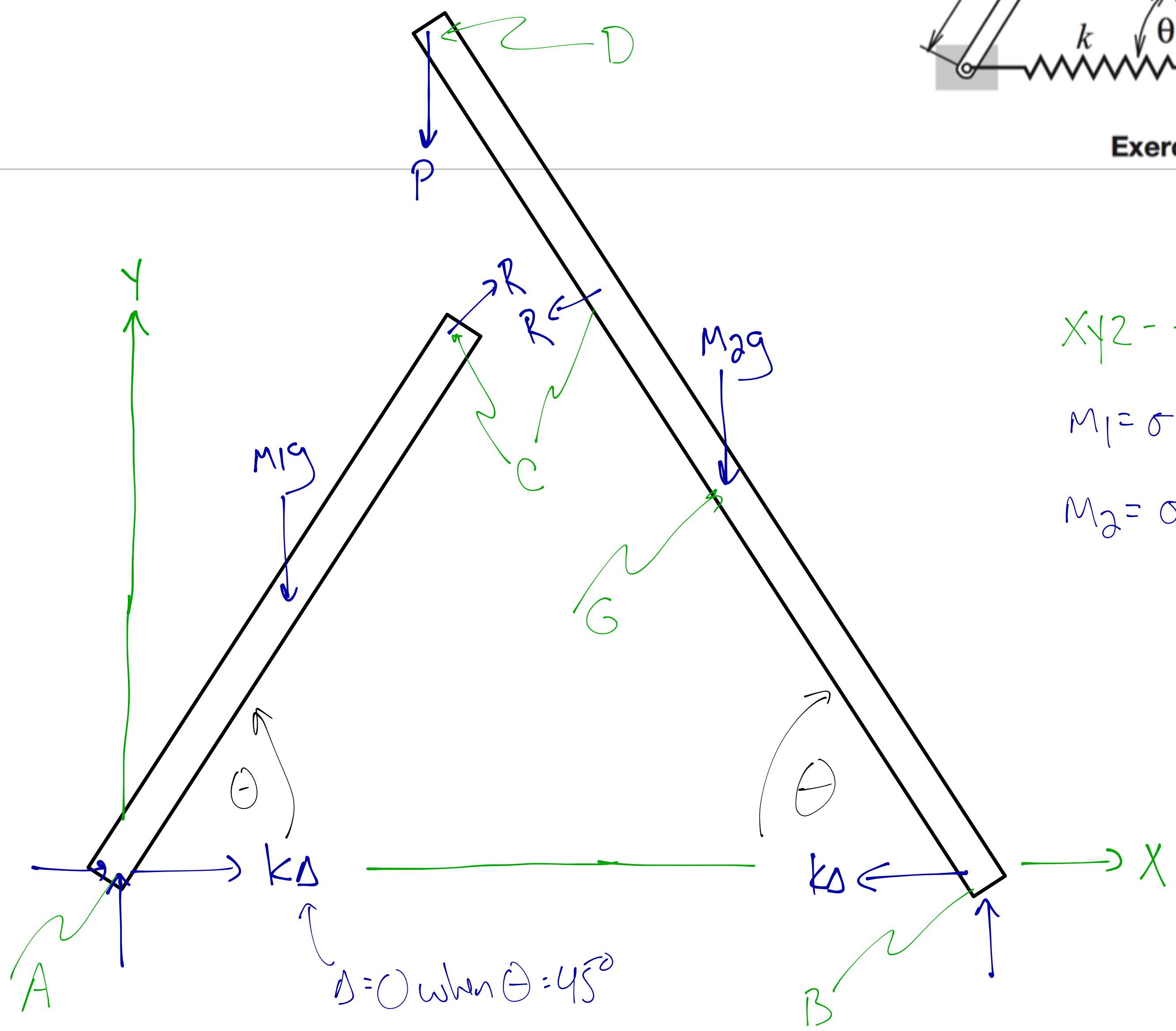
for $q_2 = \phi \rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) - \dfrac{\partial L}{\partial \phi} = 0$

# Problem 7.30

**EXERCISE 7.30** The linkage is braced by a spring of stiffness $k$ in order to support the force $P$ that acts perpendicularly to the long link. The system lies in the vertical plane, and $\sigma$ is the mass per unit length of both bars. The spring is unstretched when $\theta = 45°$. Derive the equation of motion governing $\theta$.




**Exercise 7.30**




xyz - fixed with origin at pin

$M_1 = \sigma L$

$M_2 = \sigma\left(\frac{3L}{2}\right)$

$$T = \frac{1}{2} I_A \dot{\theta} + \frac{1}{2} I_B \dot{\theta}^2 + \underbrace{\frac{1}{2} M_2 \bar{v}_G \cdot \bar{v}_G}$$

We need because long bar is not in pure rotation

$$\bar{v}_G = \bar{v}_C + (v_G)_{xyz} + \bar{\omega}_B \times \bar{r}_{G/C} = (\bar{\omega}_A \times \bar{r}_{C/A}) + (\bar{\omega}_B \times \bar{r}_{G/C})$$

Here, though

$$\bar{r}_{G/A} = \bar{r}_{C/A} + \bar{r}_{G/C} = (L\cos\theta\,\bar{i} + L\sin\theta\,\bar{j}) + \left(L - \frac{3L}{4}\right)(\cos\theta\,\bar{i} - \sin\theta\,\bar{j})$$

$$= \frac{5}{4}L\cos\theta\,\bar{i} - \frac{3}{4}L\sin\theta\,\bar{j}$$

$$\bar{v}_G = \dot{\bar{r}}_{G/A} = \dot{\theta}\left(-\frac{5}{4}L\sin\theta\,\bar{i} - \frac{3}{4}L\cos\theta\,\bar{j}\right)$$

# Problem 7.30 (cont.)

$$V = \frac{1}{2}k\Delta^2 + M_1 g h_1 + M_2 g h_2$$

$$h_1 = \frac{L}{2}\sin\theta \qquad h_2 = \frac{3L}{4}\sin\theta$$

$$\Delta = |\bar{r}_{B/A}| - \underbrace{|\bar{r}_{B/A}(\theta = 45)|}_{\text{position when } \theta = 45^\circ \text{ and spring at equil.}} = 2L\cos\theta - 2L\cos 45^\circ$$

$$L = T - V$$

$$\delta W = Q_1 \delta\theta = -P\bar{J} \cdot \delta\bar{r}_D$$

$$\bar{r}_{D/A} = \left(L\cos\theta - \frac{L}{2}\cos\theta\right)\bar{\imath} + \left(L\sin\theta + \frac{L}{2}\sin\theta\right)\bar{J}$$

$$= \frac{L}{2}\cos\theta\,\bar{\imath} + \frac{3L}{2}\sin\theta\,\bar{J}$$

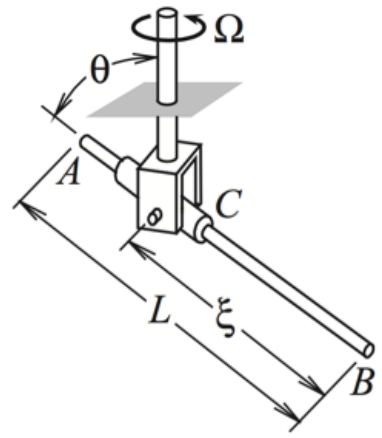
$$\bar{V}_D = -\frac{L}{2}\dot{\theta}\sin\theta\,\bar{\imath} + \frac{3L}{2}\dot{\theta}\cos\theta\,\bar{J} \rightarrow \delta\bar{r}_D = \left(-\frac{L}{2}\sin\theta\right)\delta\theta\,\bar{\imath} + \left(\frac{3L}{2}\cos\theta\right)\delta\theta\,\bar{J}$$

$$\delta W = -P\bar{J} \cdot \delta\bar{r}_D = \underbrace{-\frac{3}{2}PL\cos\theta}_{Q_1}\,\delta\theta$$

So $q_1 = \theta \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = Q_1$

# Problem 7.61

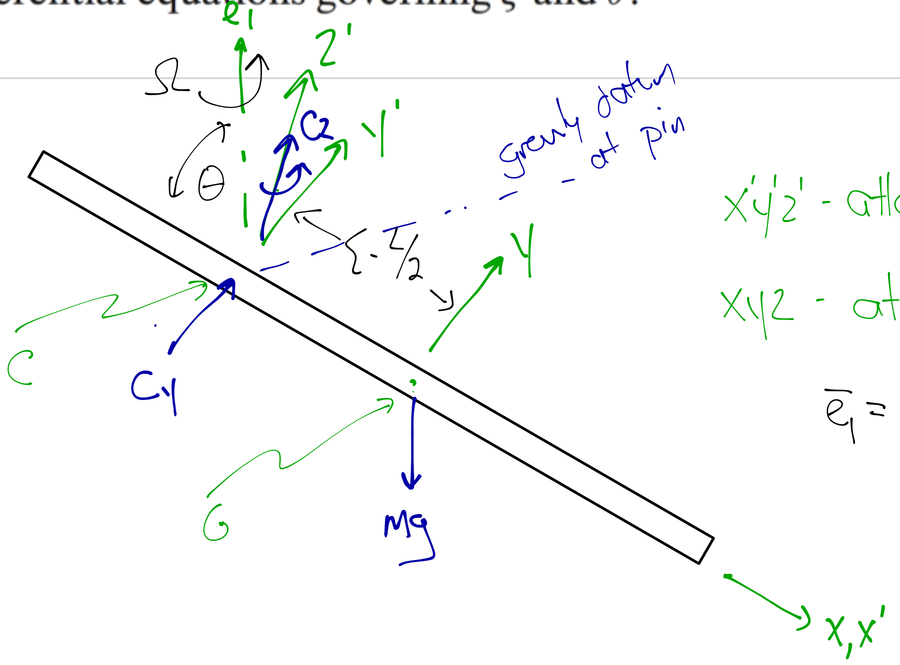
**EXERCISE 7.61** Collar $C$ is attached to the vertical shaft by a fork-and-clevis, so the angle of inclination $\theta$ of bar $AB$ is arbitrary. Because this bar slides through the collar, the distance $\xi$ from the pivot point to the end of the bar is variable, but it may be assumed that the bar does not spin about its own axis. The vertical shaft rotates at the constant rate $\Omega$. Derive the differential equations governing $\xi$ and $\theta$.




Exercise 7.61

$q_1 = \theta$

$q_2 = \xi$




x'y'z' - attached to collar $\rightarrow \omega' = \Omega \bar{e}_1 - \dot{\theta} \bar{J}$

xyz - attached to the bar $\rightarrow \bar{\omega} = \Omega e_1 - \dot{\theta} \bar{J}$

$\bar{e}_1 = -\cos\theta\, \bar{i} + \sin\theta\, \bar{k}$

$\omega = \omega' = -\Omega\cos\theta\, \bar{i} - \dot{\theta}\bar{J} + \Omega\sin\theta\, \bar{k}$

$$\bar{v}_G = (\bar{v}_G)_{x'y'z'} + \bar{\omega}' \times \bar{r}_{G/C} \qquad (\bar{v}_G)_{x'y'z'} = \dot{\xi}\,\bar{i} \qquad \bar{r}_{G/C} = \left(\xi - \frac{L}{2}\right)\bar{i}$$

$$= \dot{\xi}\,\bar{i} + \left(\xi - \frac{L}{2}\right)\dot{\theta}\,\bar{k} + \left(\xi - \frac{L}{2}\right)\Omega\sin\theta\,\bar{J}$$

$$T = \frac{1}{2} m\, \bar{v}_G \cdot \bar{v}_G + \frac{1}{2}\bar{\omega}\cdot \bar{H}_G$$

$$\bar{H}_G = [I]\bar{\omega} = I_{yy}\omega_y + I_{zz}\omega_z$$ $\leftarrow$ $I_{xx} = 0$ and all products of inertia $= 0$

$$T = \frac{1}{2} m\, \bar{v}_G \cdot \bar{v}_G + \frac{1}{2}\left(I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right)$$

$$V = mgh = -mg\left(\xi - \frac{L}{2}(\cos\theta)\right) \longrightarrow \mathcal{L} = T - V$$

No external, non-conservative forces so $Q_i = 0 \ \forall i$

So, for $q_1 = \theta \rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$

for $q_2 = \xi \rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}}\right) - \frac{\partial \mathcal{L}}{\partial \xi} = 0$