Definition of Generalized Forces (Sec. 7.4.1)

Virtual work done by real forces over virtual displacements

$$\delta W = \sum_n \bar{F}_n \cdot \delta \bar{r}_n \qquad \delta \bar{r}_n \equiv \text{virtual displacement of point where } F_n \text{ acts}$$

This is also equal to

$$\delta W = \sum_{j=1}^{N} Q_j \, \delta q_j \qquad Q_j \equiv \text{generalized forces}$$

Including moments, this expands to

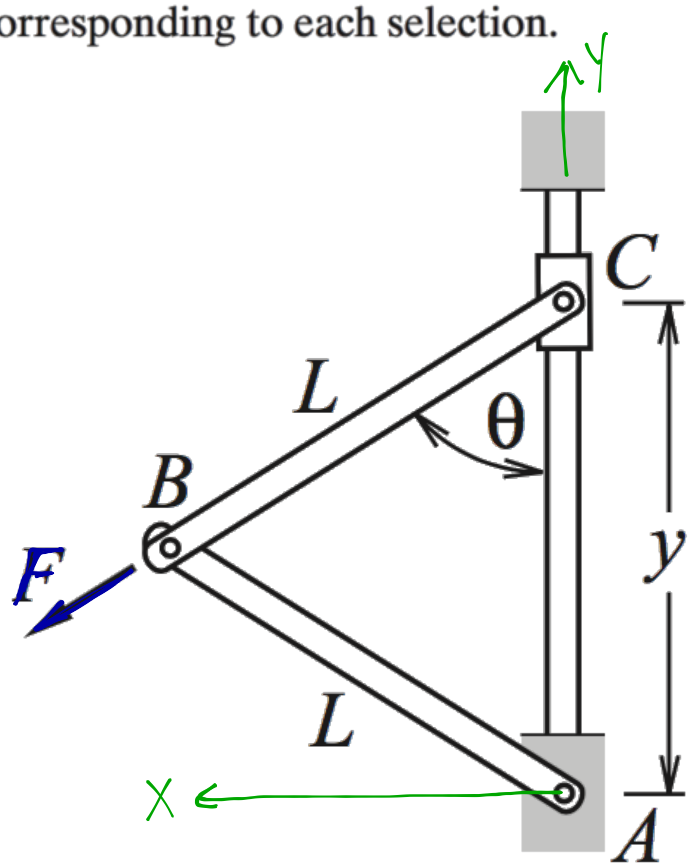
$$\delta W = \sum_n F_n \cdot \delta \bar{r}_n + \sum_k \bar{\Gamma}_k \cdot \delta \bar{\theta}_k \qquad \bar{\Gamma}_k \equiv \text{couples applied to system}$$

$$\delta \bar{\theta}_k \equiv \text{virtual rotation caused by } \bar{\Gamma}_k$$

Generalized force $\rightarrow$ component of forces and torques acting on a system that causes motion in the direction of a generalized coord.

Example 7.7

EXAMPLE 7.7 Force F acts parallel to link AB. Alternative choices for the generalized coordinate are the angle θ or the distance y. Determine the generalized force corresponding to each selection.



$$\delta W = \bar{F} \cdot \delta \bar{r}_B$$

$$\bar{r}_{B/A} = L\sin\theta\,\bar{i} + L\cos\theta\,\bar{j}$$

$$\bar{v}_B = \cancel{\bar{v}_A}^{\,0} + \cancel{(\bar{v}_B)_{xyz}}^{\,0} + \bar{\omega} \times \bar{r}_{B/A} \qquad \omega = -\dot{\theta}\,\bar{k}$$

$$= -\dot{\theta}\,\bar{k} \times \left(L\sin\theta\,\bar{i} + L\cos\theta\,\bar{j}\right)$$

$$\bar{v}_B = -L\dot{\theta}\sin\theta\,\bar{j} + L\dot{\theta}\cos\theta\,\bar{i}$$

$$\bar{v}_B = \left[L\cos\theta\,\bar{i} - L\sin\theta\,\bar{j}\right]\dot{\theta}$$

$$\delta \bar{r}_B = \left[L\cos\theta\,\bar{i} - L\sin\theta\,\bar{j}\right]\delta\theta$$

$$\bar{F} = F\sin\theta\,\bar{i} - F\cos\theta\,\bar{j}$$

$$\delta W = \bar{F} \cdot \delta \bar{r}_B = \left(F\sin\theta\,\bar{i} - F\cos\theta\,\bar{j}\right) \cdot \left(\left[L\cos\theta\,\bar{i} - L\sin\theta\,\bar{j}\right]\delta\theta\right)$$

$$= \left(\left[FL\sin\theta\cos\theta\right] + \left[FL\cos\theta\sin\theta\right]\right)\delta\theta$$

$$= 2FL\cos\theta\sin\theta\,\delta\theta$$

$$\delta W = Q_1\,\delta q_1 \qquad q_1 = \theta \qquad \text{so} \qquad \boxed{Q_1 = 2FL\cos\theta\sin\theta}$$

Conservative Forces (Sec. 7.4.3)

For conservative forces work done = change in potential energy

$$W_{1\to 2} = U_1 - U_2$$

$W_{1\to 2}$ = work from state 1 to state 2

U_1 = potential energy at state 1

U_2 = potential energy at state 2

Potential energy depends on position and time, so write as a function of gen. coord + time

$$U(q_j, t)$$

Find that the generalized force from conservative forces is:

$$(Q_j)_{conser.} = -\frac{\partial}{\partial q_j} U(q_j, t)$$

But not all forces are conservative (of course) so total gen. force becomes

$$(Q_j)_{Total} = \underbrace{Q_j}_{} - \frac{\partial}{\partial q_j} U(q_j, t)$$

generally used to describe all generalized forces not described by potential energy

Note: If we're not sure if a force is conservative, treat it as nonconservative (eval via virtual work not potential energy)

Lagrange's Equations (Sec. 7.5)

See book for derivation working from a single particle

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j \qquad j=1,\ldots,N$$

N: # of generalized coordinates
T: total kinetic energy
V: total potential energy

- result is N eq. of motion for N gen. coords
- In this form, any constraint forces show up in Q_j

For now, let's only look at holonomic systems (no constraint forces in Q_j)

We can also write using the Lagrangian, L

$$L = T - V$$

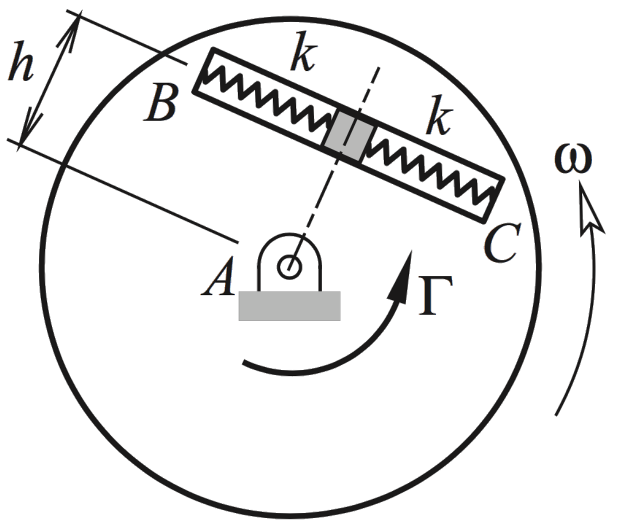
Since potential energy does not depend on $\dot{q}_j$,

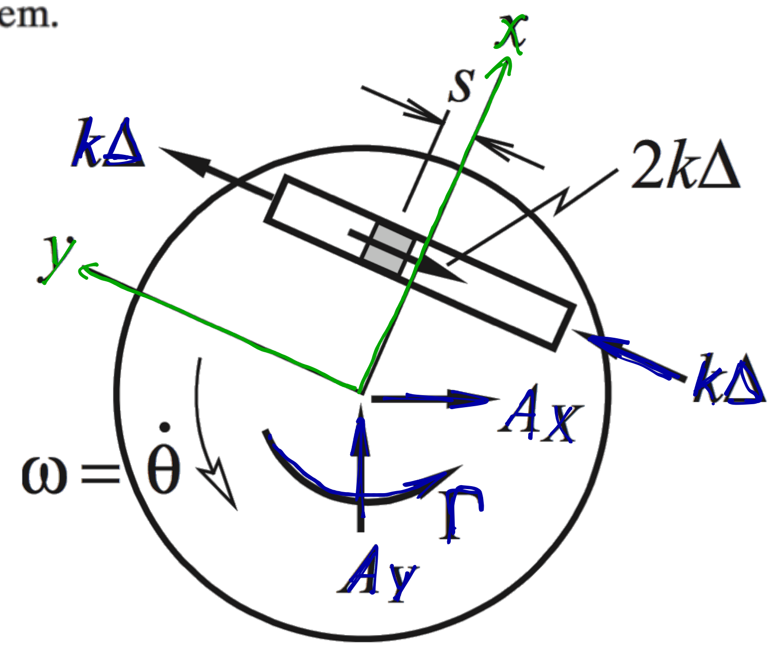
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad j=1,2,\ldots,N$$

Be careful with partial and total derivatives

Example 7.13

EXAMPLE 7.13 The table rotates in a horizontal plane about bearing A due to a torque Γ whose time dependence is known. The mass of the table is m_1, and its radius of gyration about its center is κ. The slider, whose mass is m_2, moves within groove BC under the restraint of two springs that are unstretched in the position shown. Derive the equations of motion for this system.





Q: What should we choose as generalized coordinates?

$\bar{q} = (\theta, s)$ $\quad$ s ≡ displacement of mass from eq. length of springs

Q: What generalized forces are there?

Γ is only nonconservative force. It acts directly on θ so $\delta W = \Gamma \delta\theta \longrightarrow Q_1 = \Gamma \quad Q_2 = 0$

Q: What is the kinetic energy?

$$T = \underbrace{\tfrac{1}{2}(I_{zz})_{table}\,\omega^2}_{\text{Table in pure rotation}} + \underbrace{\tfrac{1}{2} m \bar{v}_s \cdot \bar{v}_s}_{\text{point mass total vel term}}$$

Q: What is $(I_{zz})_{table}$?

$(I_{zz})_{table} = Mk^2$ $\quad$ k ≡ radius of gyration given in problem

Q: What is v_s?

$$\bar{v}_s = \cancel{\bar{v}_A}^{\,0} + (\bar{v}_s)_{xyz} + \bar{\omega} \times \bar{r}_{s/A} \qquad \bar{r}_{s/A} = l\bar{i} + s\bar{j} \qquad (\bar{v}_s)_{xyz} = \dot{s}\bar{j} \qquad \bar{\omega} = \dot{\theta}\bar{k}$$

$$= \dot{s}\bar{j} + \left(\dot{\theta}k \times (l\bar{i} + s\bar{j})\right) = -s\dot{\theta}\bar{i} + (\dot{s} + l\dot{\theta})\bar{j}$$

Example 7.13 (cont.)

So

$$T = \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}m\left[(s\dot{\theta})^2 + (\dot{s}+l\dot{\theta})^2\right]$$

Q: What is the potential energy?

$$V = \frac{1}{2}k\Delta_1^2 + \frac{1}{2}k\Delta_2^2 \qquad \Delta = \Delta_1 = \Delta_2 \longrightarrow V = k\Delta^2$$

We defined s along the spring axis and as the deflection from spring eq., so

$$V = ks^2$$

$$L = \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}m\left[(s\dot{\theta})^2 + (\dot{s}+l\dot{\theta})^2\right] - ks^2$$

Now, apply Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad j = 1, \dots, M$$

$\leftarrow$ Here $M=2$ and $\bar{q} = (\theta, s)$ $\Big\}$ 2 equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_1 \qquad \underline{\text{and}} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = Q_2$$

For $q_1 = \theta$

$$\frac{\partial L}{\partial \dot{\theta}} = Mk^2\dot{\theta} + \frac{1}{2}m\left[2s^2\dot{\theta} + 2(\dot{s}+l\dot{\theta})l\right] = Mk^2\dot{\theta} + ms^2\dot{\theta} + ml\dot{s} + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = Mk^2\ddot{\theta} + ms^2\ddot{\theta} + 2ms\dot{s}\dot{\theta} + ml\ddot{s} + ml^2\ddot{\theta} = (Mk^2 + ms^2 + ml^2)\ddot{\theta} + 2ms\dot{s}\dot{\theta} + ml\ddot{s}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\boxed{(Mk^2 + ms^2 + ml^2)\ddot{\theta} + 2ms\dot{s}\dot{\theta} + ml\ddot{s} = \Gamma}$$

For $q_2 = s$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} + ml\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s} + ml\ddot{\theta} \qquad \boxed{m\ddot{s} + ml\ddot{\theta} - ms\dot{\theta}^2 + 2ks = 0}$$

$$\frac{\partial L}{\partial s} = ms\dot{\theta}^2 - 2ks$$