# Constraint Equations (Sec. 7.2.2)

select $N$ generalized coords., $q_i$ = $j^{th}$ generalized coordinate

Write the constraints on the system in functional form:

$$f_i(q_n, t) = 0 \quad \longleftarrow \quad \text{configuration constraint}$$

We can also constrain velocity (linear or angular):

$$\dot{f}_i = \sum_{j=1}^{N} \left[ \frac{\partial}{\partial q_j} f_i(q_n, t) \right] \dot{q}_j + \frac{\partial}{\partial t} f_i(q_n, t) = 0 \quad \longleftarrow \quad \text{velocity constraint}$$

It is often easier to specify velocity constraints than config. constraints (that are valid for all time)

Often, we'll write a more general form of the velocity constraint

$$\sum_{j=1}^{N} a_{ij}(q_n, t)\dot{q}_j + b_i(q_n, t) = 0 \quad \longleftarrow \quad \text{linear velocity constraint}$$

($\uparrow$ mathematically linear, not necessarily straight line)

Key insight:

config constraint is algebraic/transcendental eq. governing how arrived at current state

velocity constraint is restriction on how a system can move given its current state, representing kinematically admissible motion

Pfaffian form: mult. by $dt$

$$\sum_{j=1}^{N} a_{ij}(q_n, t)\, dq_j + b_i(q_n, t)\, dt = 0$$

— restricts motion in given interval $dt$

# Holonomic vs nonholonomic constraints

Pfaffian form of velocity and config constraints only differ by a multi factor

This integrating factor (since config is integral of vel.) can be a function of gen coord & time.

So, a given velocity constraint is derivable from a config constraint if:

$$g_i(q_n,t)\, a_{ij}(q_n,t) = \frac{\partial}{\partial q_i} f_i(q_n,t) \qquad j=1,\dots,N$$

$$g_i(q_n,t)\, b_i(q_n,t) = \frac{\partial}{\partial t} f_i(q_n,t)$$

where $g_i(q_1,\dots q_N,t)$ is the integrating factor

If this is satisfied for all $j$ $\rightarrow$ constraint is holonomic

else $\rightarrow$ nonholonomic

Note: $\frac{\partial f_i}{\partial q_j}$ terms are a Jacobian set of holonomic constraints

So $a_{ij}$ coefficients are called Jacobian Constraint Matrix

(even when constraint is nonholonomic)

Note: Determining if a velocity constraint is holonomic is nontrivial

# Holonomic vs nonholonomic constraints (cont.)

Nonholonomic constraints also dictate how many generalized coords we need to use:

If $J_c \equiv$ # configuration constraints

$J_v \equiv$ # of nonholonomic velocity constraints

$D =$ # of DOF

$N = D + J_c + J_v \longrightarrow$ in general minimum set becomes $N = D + J_v$

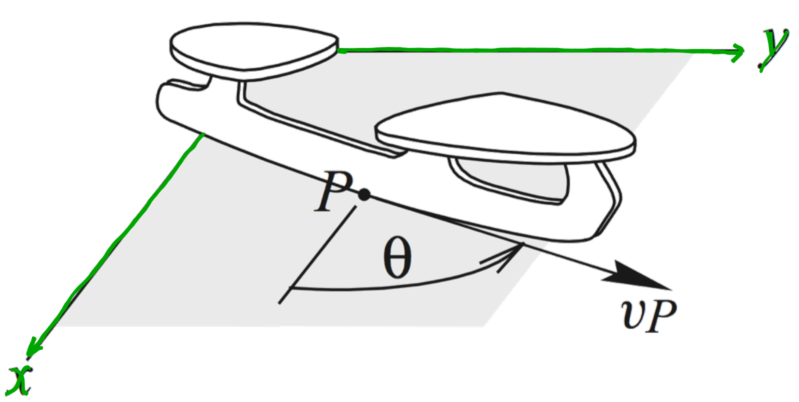
b/c can choose coords to allow calc. of $J_c$

But if nonholonomic, we can't diff, so need to know motion

Key point:

So, generally treat velocity constraints as nonholonomic

As an example consider the ice skate:




3 DOF ($x, y, \theta$ one choice)

However, $v_P$ must align with blade

$$\bar{v}_P = \dot{x}_P \bar{\imath} + \dot{y}_P \bar{\jmath} = v_P(\cos\theta\,\bar{\imath} + \sin\theta\,\bar{\jmath})$$

$$\dot{x}_P = v_P \cos\theta \rightarrow v_P = \frac{\dot{x}_P}{\cos\theta}$$

$$\dot{y}_P = v_P \sin\theta \qquad \dot{y}_P = \frac{\dot{x}_P}{\cos\theta}\sin\theta \rightarrow \underbrace{\dot{x}_P \sin\theta - \dot{y}_P \cos\theta = 0}$$

Matches standard velocity constraint form

$a_{11} = \cos\theta \quad a_{12} = -\sin\theta$

$a_{13} = 0 \quad$ and $\quad b_1 = 0$

$\Big\}$ nonholonomic

Constraint means that

$$\underbrace{dy_P = dx_P \tan\theta}_{\text{restricts new position of P}} \leftarrow \text{Pfaffian form of}$$

Degrees of Freedom (for nonholonomic) $\rightarrow$ $D = N - J_c - J_v$

# Configuration Space (Sec. 7.2.3)

Alternate way to describe position of system. Ginsberg uses it to briefly introduce the foundations of variational methods. Please read.

# Evaluation of Virtual Displacements (Sec. 7.3)

## Analytical Method (Sec. 7.3.1)

Please read. We will not cover in lecture.

## Kinematical Method (Sec. 7.3.2)

Recognize relationship between virtual and real displacements

Use velocity formulations to determine virtual displacements

Imagine arbitrary point A in a system. Its velocity is:

$$\bar{v}_A = \frac{d\bar{r}_A}{dt} = \sum_{j=1}^{M} \frac{\partial r_A}{\partial q_j} \dot{q}_j + \frac{\partial r_A}{\partial t}$$

So, the actual displacement of A over small time interval dt is:

$$\left[\bar{v}_A = \frac{d\bar{r}_A}{dt} = \sum_{j=1}^{M} \frac{\partial r_A}{\partial q_j} \dot{q}_j + \frac{\partial r_A}{\partial t}\right] dt \longrightarrow \underbrace{d\bar{r}_A = \bar{v}_A dt = \sum_{j=1}^{M} \frac{\partial r_A}{\partial q_j} dq_j + \frac{\partial r_A}{\partial t} dt}$$

Use this to write virtual displacement

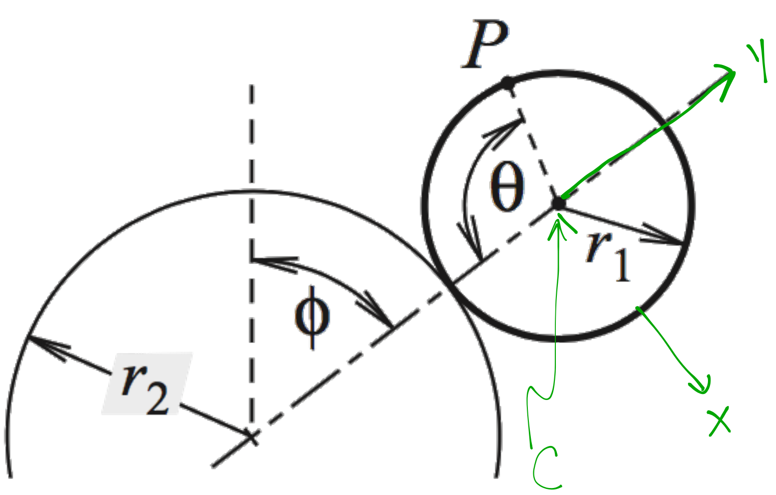
$$\delta \bar{r}_A = \sum_{j=1}^{M} \frac{\partial r_A}{\partial q_j} \delta q_j \longleftarrow$$ No dt term b/c no time passes

If a system is time dependent, drop all terms that do not contain a generalized velocity

In other words, write the velocity of the point and replace all $\frac{d\bar{q}_i}{dt}$ terms with $\delta \bar{q}_i$.

# Example 7.5

**EXAMPLE 7.5** The disk rolls without slipping over the stationary cylinder. The rotation $\phi$ of the line of centers from vertical is selected as the generalized coordinate for the disk. Point $P$ contacted the cylinder when $\phi = 0$. Determine the virtual displacement of this point.




Because rolling (arc lengths are equal)

$$r_1\theta = r_2\phi \rightarrow \theta = \frac{r_2}{r_1}\phi$$

Know $v_c$ follows a circular path

$$\bar{v}_c = \omega\bar{k} \times r_1\bar{j} = -r_1\omega\bar{i} = (r_1+r_2)\dot{\phi}\bar{i}$$

$$\omega = -\left(\frac{r_1+r_2}{r_1}\right)\dot{\phi}$$

So

$$\bar{v}_P = \bar{v}_c + \cancel{(\bar{v}_P)_{xyz}}^{\,0} + \bar{\omega} \times \bar{r}_{P/C}$$

$$\bar{r}_{P/C} = r_1\left(-\sin\theta\,\bar{i} - \cos\theta\,\bar{j}\right) = r_1\left[-\sin\left(\frac{r_2}{r_1}\phi\right)\bar{i} - \cos\left(\frac{r_2}{r_1}\phi\right)\bar{j}\right]$$

$$v_P = (r_1+r_2)\dot{\phi}\bar{i} + \left[-\frac{r_1+r_2}{r_1}\dot{\phi}\right]\left(r_1\cos\left(\frac{r_2}{r_1}\phi\right)\bar{i} - r_1\sin\left(\frac{r_2}{r_1}\phi\right)\bar{j}\right)$$

Now, replace $v_P$ with $\delta\bar{r}_P$ and $\dot{q}_i$ with $\delta q_i$

$$\boxed{\delta\bar{r}_P = (r_1+r_2)\,\delta\phi\,\bar{i} + \left[-(r_1+r_2)\cos\left(\frac{r_2}{r_1}\phi\right)\bar{i} + (r_1+r_2)\sin\left(\frac{r_2}{r_1}\phi\right)\bar{j}\right]\delta\phi}$$

Please look at Example 7.6. It nicely outlines some of the differences that come with the choice of generalized coordinates.