______ **Problem 1 – 40 Points**

In Figure 1, a disk of radius R spins about link BC at a rate of $\dot{\phi}$. Link ABC rotates about axis Z at a constant rate, $\dot{\psi}$. The angle from link AB from vertical, θ, is also variable. The angle of the bend in the ABC linkage, β, is fixed.

a. Write the angular velocity and angular acceleration of the disk. For each, be sure to indicate how to resolve all the components into the same frame.

b. What is the velocity of point D?

c. What is the acceleration of point D?

Let's extend this problem into a Ch. 6 example by adding a few forces/torques and writing the equations of motion.

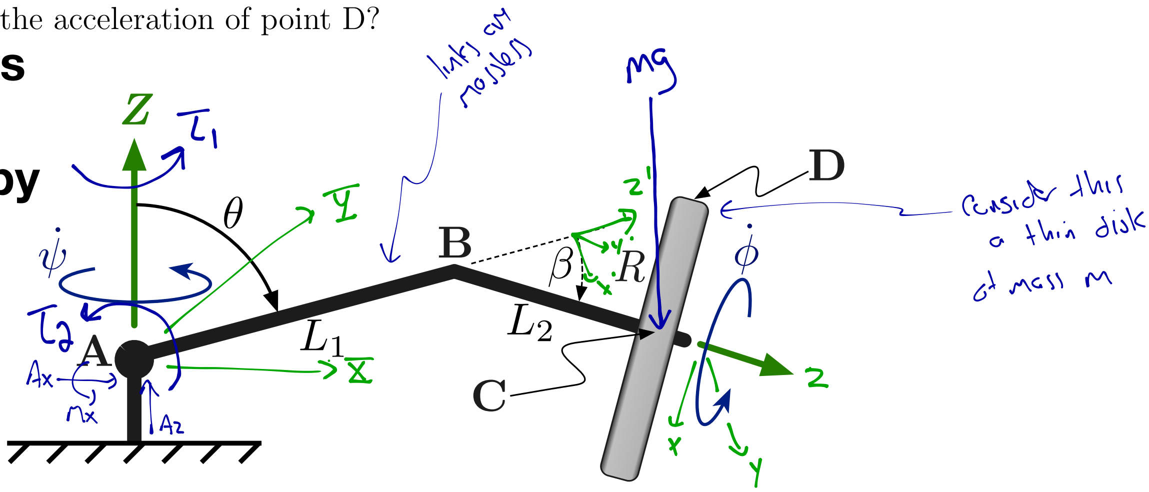


Figure 1: A Spinning disk on a Bent Linkage

a. Define coord. frames

$\bar{X}\bar{Y}\bar{Z}$ – fixed to AB, but $\bar{X}$ always horizontal

$x'y'z'$ – fixed to AB, with z' along AB

xyz – fixed to BC, with z along BC

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ k' \end{bmatrix} = R_\theta \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} \rightarrow R_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ k \end{bmatrix} = R_\beta \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ k' \end{bmatrix} = \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ k' \end{bmatrix} \rightarrow R_\beta = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

Problem 1 (cont.)

a.

$$R_\beta R_\Theta = \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix}\begin{bmatrix} c\Theta & 0 & -s\Theta \\ 0 & 1 & 0 \\ s\Theta & 0 & c\Theta \end{bmatrix} = \begin{bmatrix} c\beta c\Theta - s\beta s\Theta & 0 & -c\beta s\Theta - s\beta c\Theta \\ 0 & 1 & 0 \\ s\beta c\Theta + c\beta s\Theta & 0 & -s\beta s\Theta + c\beta c\Theta \end{bmatrix}$$

so $\bar{k} = (s\beta c\Theta + c\beta s\Theta)\bar{I} + (-s\beta s\Theta + c\beta c\Theta)\bar{K}$

$$\bar{\omega}_{disk} = \dot{\psi}\bar{K} + \dot{\Theta}\bar{J} + \dot{\phi}\bar{k} = \dot{\psi}\bar{K} + \dot{\Theta}\bar{J} + R_\beta R_\Theta \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$\boxed{\bar{\omega}_{disk} = \dot{\phi}(s\beta c\Theta + c\beta s\Theta)\bar{I} + \dot{\Theta}\bar{J} + \left[\dot{\phi}(-s\beta s\Theta + c\beta c\Theta) + \dot{\psi}\right]\bar{K}}$$

$$\bar{\alpha}_{disk} = \cancel{\ddot{\psi}\bar{K}}^{\,0} + \cancel{\dot{\psi}\dot{\bar{K}}}^{\,0} + \ddot{\Theta}\bar{J} + \dot{\Theta}\dot{\bar{J}} + \ddot{\phi}\bar{k} + \dot{\phi}\dot{\bar{k}}$$

$$= \ddot{\Theta}\bar{J} + \dot{\Theta}(\bar{\omega}_{XYZ} \times \bar{J}) + \ddot{\phi}\bar{k} + \dot{\phi}(\bar{\omega}_{xyz} \times \bar{k})$$

where $\bar{\omega}_{XYZ}$ is angular velocity of XYZ frame – $\bar{\omega}_{XYZ} = \dot{\psi}\bar{K}$

ω_{xyz} is angular velocity of xyz frame – $\bar{\omega}_{xyz} = \dot{\psi}\bar{K} + \dot{\Theta}\bar{J}$

$$\bar{\alpha}_{disk} = \ddot{\Theta}\bar{J} + \dot{\Theta}(\dot{\psi}\bar{K} \times \bar{J}) + \ddot{\phi}\bar{k} + \dot{\phi}\left[(\dot{\psi}\bar{K} + \dot{\Theta}\bar{J}) \times \bar{k}\right]$$

$$\boxed{\bar{\alpha}_{disk} = \ddot{\Theta}\bar{J} + (-\dot{\Theta}\dot{\psi}\bar{I}) + \ddot{\phi}\bar{k} + \dot{\phi}\left[(\dot{\psi}\bar{K} + \dot{\Theta}\bar{J}) \times \bar{k}\right]}$$

← see above for $\bar{k}$ in terms of $\bar{I}$ and $\bar{K}$ to allow this cross product

See the Jupyter Notebook for the completed expression

Problem 1 Extension

Write the equations of motion.

Q: How many DOF?... so how many eq. of motion?

$3 \rightarrow \psi, \theta, \phi$

Q: Where should we sum moments about?

Use point C.. It's the COM of the disk and will let us ignore gravity in the moment eq.

So,

$$\Sigma \bar{M}_c = \dot{\bar{H}}_c$$

$$\Sigma \bar{M}_c = \tau_1 \bar{K} + \tau_2 \bar{J} + M_x \bar{I} + \left(\bar{r}_{A/C} \times A_x \bar{I} + A_z \bar{K} \right)$$

$$= \tau_1 \bar{K} + \tau_2 \bar{J}' + M_x \bar{I} + \left(\bar{r}_{A/C} \times A_x \bar{I} + A_z \bar{K} \right)$$

Q: What frame should we use to write this equation?

If we use xyz, then the axes are principle. The other frames require manipulation of the inertia properties.

This means we could use Euler's Eq if we write $\bar{\omega}_{disk}$ and $\bar{\alpha}_{disk}$ in this frame!

So, we'd need to do a coord transformation on $\bar{\omega}_{disk} + \bar{\alpha}_{disk}$ to xyz and write $\Sigma \bar{M}_c$ in that frame

Q: What about $\Sigma \bar{F}$

$\Sigma \bar{F} = m\bar{a}_c$ ⟵ we found $\bar{a}_c$ in part c of the problem

$$\Sigma \bar{F} = A_x \bar{I} + (A_z - mg) \bar{K}$$

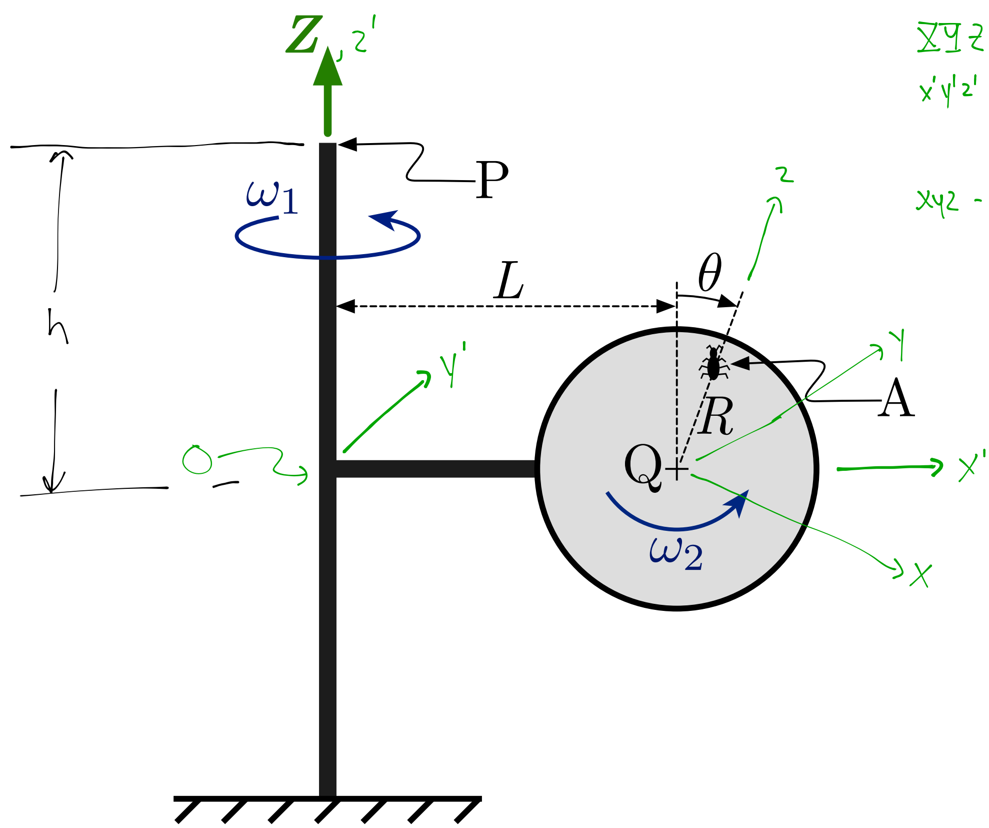
______ **Problem 2 – 30 Points**

Figure 2 shows an instantaneous view of a disk rotating at a rate ω_2 about point Q. An ant is relaxing on the disk (*i.e.* its location on the disk is fixed) at a distance R from its center. When, $\theta = 0$, the ant reaches her topmost position. The center of the disk is located on the T-bar at a distance L from the axis about which the bar is spinning, Z. The rate of rotation of the T-bar is described by ω_1.

a. What is the velocity of the ant, $\bar{v}_A$?

b. What is the acceleration of the ant, $\bar{a}_A$?

c. What does the ant observe as the velocity and acceleration of point P, the top of the T-bar?



XYZ · fixed with Z up

x'y'z' - fixed to T bar with z' up and x' along T

xyz - fixed to disk with z from Q to A

Figure 2: An Ant Riding on a Spinning Disk on a T-bar

a. $\bar{V}_A = \bar{V}_Q + \bar{\omega}_{xyz} \times \bar{r}_{A|Q}$ $\quad \bar{\omega}_{xyz} = \omega_1 \bar{k}' - \omega_2 \bar{J} = \omega_1 \bar{k}' - \omega_2 \bar{J}'$ $\quad \bar{r}_{A|Q} = R\bar{k}$

$\bar{V}_Q = \cancel{\bar{V}_O}^{\,0} + \bar{\omega}_{x'y'z'} \times \bar{r}_{Q|O}$ $\quad \bar{r}_{Q|O} = L\bar{i}'$ $\quad \bar{\omega}_{x'y'z'} = \omega_1 \bar{k}'$

$\bar{V}_Q = \omega_1 \bar{k}' \times L\bar{i}' = L\omega_1 \bar{J}'$

$$\begin{Bmatrix} \bar{i} \\ \bar{J} \\ \bar{k} \end{Bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{Bmatrix} \bar{i}' \\ \bar{J}' \\ \bar{k}' \end{Bmatrix}$$

So

$\bar{V}_A = (L\omega_1 \bar{J}') + (-\omega_2 \bar{J}' + \omega_1 \bar{k}') \times R\bar{k}$

$\bar{V}_A = (L\omega_1 \bar{J}') + (-\omega_2 \bar{J}' + \omega_1 \bar{k}') \times (R\sin\theta\, \bar{i}' + R\cos\theta\, \bar{k}') \leftarrow \bar{k} = s\theta\, \bar{i}' + c\theta\, \bar{k}'$

Problem 2 Extension

What is the angular momentum of the disk?

Q: Where should we calculate this?

Let's use point Q, it's the COM. We can also see that xyz are principle axes for the disk.

So

$$\bar{H}_Q = I\bar{\omega} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \bar{\omega}_{disk}$$

we need to write this in the xyz frame

$$\bar{\omega}_{disk} = \bar{\omega}_{xyz} = \omega_1 \bar{k}' - \omega_2 \bar{j}'$$

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = \underbrace{\begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}}_{R_\theta} \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} = \underbrace{\begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}}_{R_\theta^T} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix}$$

$$\bar{\omega}_{disk} = \omega_1(-s\theta\bar{i} + c\theta\bar{k}) - \omega_2(\bar{j})$$

$$= -\omega_1 \sin\theta\, \bar{i} - \omega_2 \bar{j} + \omega_1 \cos\theta\, \bar{k}$$

So

$$\bar{H}_Q = -I_{xx}\omega_1 \sin\theta\, \bar{i} - I_{yy}\omega_2 \bar{j} + I_{zz}\omega_1 \cos\theta\, \bar{k}$$

Q: How would we get the equations of motion?

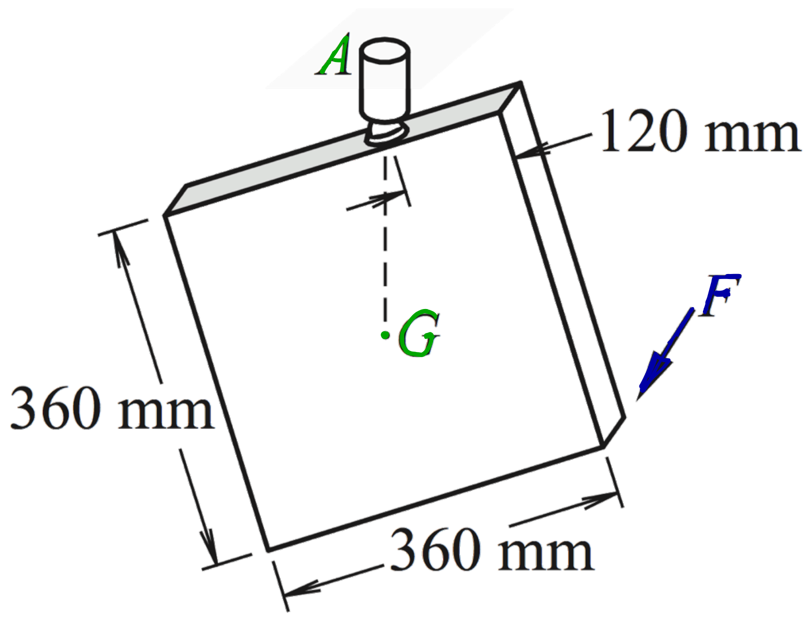
$$\Sigma \bar{M}_Q = \dot{\bar{H}}_Q \quad \text{and} \quad \Sigma \bar{F} = m\bar{a}_Q$$

$\bar{a}_Q \leftarrow$ pure rotation at radius L about point O

recognizing this will simplify the problem

Example 6.11

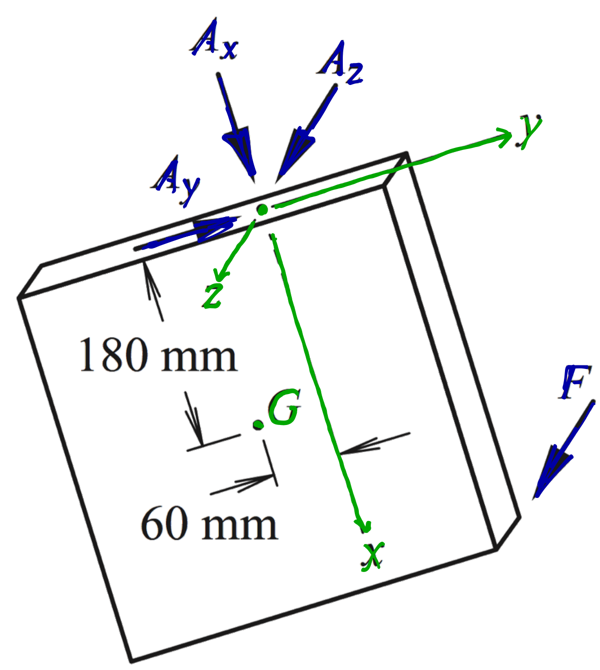
EXAMPLE 6.11 A 10-kg square plate suspended by ball-and-socket joint A is at rest when it is struck by a hammer. The impulsive force $\bar{F}$ generated by the hammer is normal to the surface of the plate, and its average value during the 4-ms interval that it acts is 5000 N. Determine the angular velocity of the plate at the instant following the impact and the average reaction at the support.



F >> mg so ignore gravity

Sum moments about A (ignore moments from reaction)

Define Inertia prop. about A



Initial velocity – $v_1 = 0$

Initial angular velocity – $\omega_1 = 0$

Find angular velocity – $\omega_2 = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}$

Velocity of com is then:

$$\bar{v}_G = \cancel{\bar{v}_A}^{\,0} + \cancel{(\bar{v}_G)_{xyz}}^{\,0} + \bar{\omega}\times\bar{r}_{G/A} \qquad \bar{r}_{G/A} = 0.18\bar{i} - 0.06\bar{j}$$

$$= (\omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}) \times (0.18\bar{i} - 0.06\bar{j})$$

$$= -0.06\omega_x\bar{k} - 0.18\omega_y\bar{k} + 0.18\omega_z\bar{j} + 0.06\omega_z\bar{i} = 0.06\omega_z\bar{i} + 0.18\omega_z\bar{j} + (-0.06\omega_x - 0.18\omega_y)\bar{k}$$

Find angular momentum at time 2

Rot. Impulse/Mom

$(H_A)_2$ ← Get $(H_A)_2$ from angular velocity at time 2 ← Equate components from these

It also $= \Sigma\bar{M}_A\,\Delta t$

$$\Sigma\bar{M}_A = (0.36\bar{i} + 0.12\bar{j}) \times F k \qquad \Delta t = 0.004\,s$$

$$m(\bar{v}_2) = \Sigma\bar{F}\,\Delta t$$

$$\Sigma\bar{F} = A_x\bar{i} + A_y\bar{j} + (A_z + F)\bar{k}$$

Use these to solve for Reaction forces

Q: What direction with A_z be?

Same as $\bar{F}$!