Chapter 6 - Newton-Euler Equations of Motion

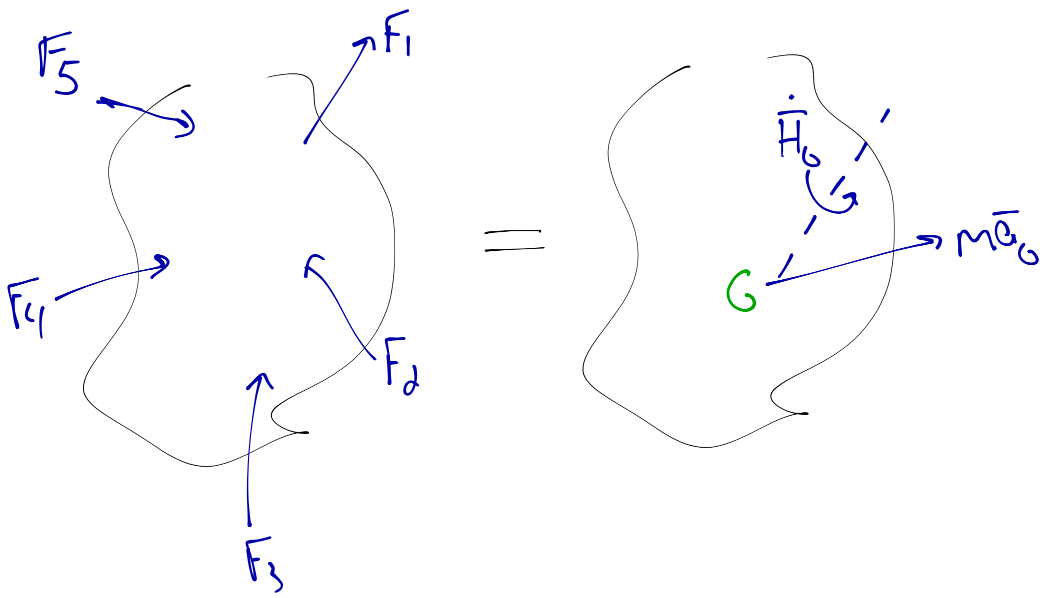
Fundamental Equations (Sec. 6.1)

$$\sum \bar{F} = m\bar{a}_G$$

$$\sum \bar{M}_A = \frac{\partial \bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A$$

$\longleftarrow$ A must be 1) COM

2) $\bar{a}_A = 0$ (pure rotation about A)



These are <u>vector</u> equations, so they require a frame.

For now, use body-fixed frames.

We can, of course, write the components in each coord by taking dot product with that coord.

$$\sum \bar{F} \cdot \hat{i} = m\bar{a}_G \cdot \hat{i} \quad , \quad \sum \bar{F} \cdot \hat{j} = m\bar{a}_G \cdot \hat{j} \quad , \quad \sum \bar{F} \cdot \hat{k} = m\bar{a}_G \cdot \hat{k}$$

Accel is often written in a global coord sys... here $\hat{x}\hat{y}\hat{z}$

$$\sum \bar{M}_A \cdot \bar{i} = \left(\frac{\partial \bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A \right) \cdot \bar{i} \quad , \quad \sum \bar{M}_A \cdot \bar{j} = \left(\frac{\partial \bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A \right) \cdot \bar{j}$$

$$\sum \bar{M}_A \cdot \bar{k} = \left(\frac{\partial \bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A \right) \cdot \bar{k}$$

We can, of course also write these in matrix notation:

Fundamental Equations (cont.)

We can, of course also write those in matrix notation:

Remember that:

$$\bar{H}_A = \left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right)\bar{i} + \left(I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z\right)\bar{j} + \left(I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y\right)\bar{k}$$

and

$$\frac{\partial H_A}{\partial t} = \left(I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z\right)\bar{i} + \left(I_{yy}\alpha_y - I_{yx}\alpha_x - I_{yz}\alpha_z\right)\bar{j} + \left(I_{zz}\alpha_z - I_{zx}\alpha_x - I_{zy}\alpha_y\right)\bar{k}$$

then

$$\dot{\bar{H}}_A = \bar{I}\bar{\alpha} + \bar{\omega} \otimes \bar{I}\bar{\omega} = \bar{I}\begin{bmatrix}\alpha_x \\ \alpha_y \\ \alpha_z\end{bmatrix} + \begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix} \otimes \left[\bar{I}\begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix}\right] = \begin{bmatrix}\sum \bar{M}_A \cdot \bar{i} \\ \sum \bar{M}_A \cdot \bar{j} \\ \sum \bar{M}_A \cdot \bar{k}\end{bmatrix}$$

Looking at the components of this equation, we can find Euler's Eq of Motion if the body-fixed axes are principal

$$\sum \bar{M}_A \cdot \bar{i} = I_{xx}\alpha_x - \left(I_{yy} - I_{zz}\right)\omega_y\omega_z$$

$$\sum \bar{M}_A \cdot \bar{j} = I_{yy}\alpha_y - \left(I_{zz} - I_{xx}\right)\omega_x\omega_z$$

$$\sum M_A \cdot \bar{k} = I_{zz}\alpha_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y$$

Remember that $\dot{\bar{H}}_A$ can be nonzero even if rotation rates are constant.

Q: What does this mean for the moments acting on the system?

There need to be some → Gyroscopic Moments (balance $\omega_i\omega_j$ product terms above)

Steps to Forming (Newton-Euler) Equations of Motion (Sec 6.1.2)

1. Draw a Free Body Diagram for all bodies with significant mass. Be sure to include all constraint forces.

2. Choose about which point to sum moments. It should be the COM or have zero acceleration (pure rotation).

3. Attach a body-fixed frame to the body with origin at point A. Choose axes orientation to make determining the inertia properties easy. It's good practice to include this coordinate system on the FBD.

4. Write all known information (specified functions of time, etc) in equation form.

5. Determine the angular velocity and angular acceleration of the body in the body-fixed frame. Be sure to include any/all kinematic constraints.

6. Determine the inertia properties with respect to the body-fixed frame.

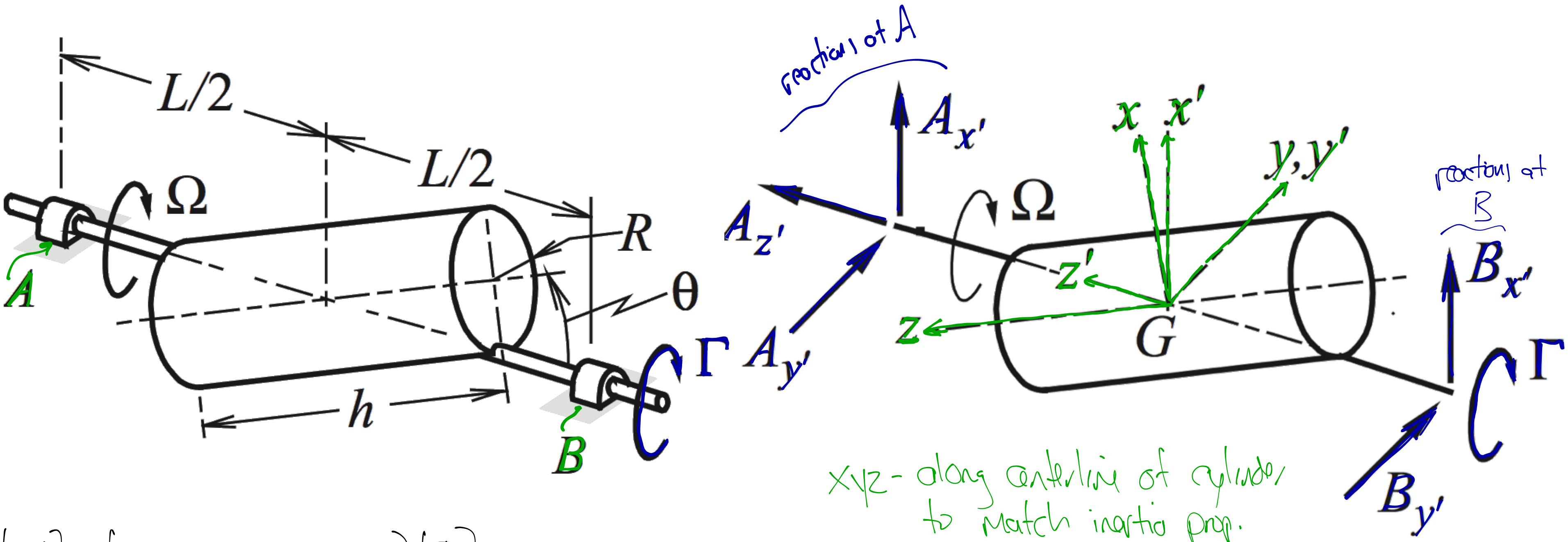
7. Compute moment about point A and equate it to the total time derivative of the angular momentum.

8. If needed, form the force equations about the body COM.

9. Check that the proper number of equations of motion resulted.

Example 6.1

EXAMPLE 6.1 The cylinder, whose mass is m, is welded to the shaft such that its center is situated on the axis of rotation. Application of a constant torque Γ at $t = 0$ causes the rotation rate Ω to increase from zero. Derive expressions for Ω and the reactions at bearings A and B as functions of the elapsed time.



xyz - along centerline of cylinder to match inertia prop. in lookup table

x'y'z' - aligned z' with shaft

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix}$$

$\bar{\omega} = \Omega \bar{k}' = \Omega \sin\theta \,\bar{i} + \Omega \cos\theta \,\bar{k}$ $\leftarrow$ $\omega_x = \Omega \sin\theta$, $\omega_y = 0$, and $\omega_z = \Omega \cos\theta$

$\bar{\alpha} = \dot{\Omega} \bar{k}' = \dot{\Omega} \sin\theta \,\bar{i} + \dot{\Omega} \cos\theta \,\bar{k}$ $\leftarrow$ $\alpha_x = \dot{\Omega} \sin\theta$, $\alpha_y = 0$, and $\alpha_z = \dot{\Omega} \cos\theta$

From lookup table, find $I_{xx} = I_{yy} = m\left(\frac{1}{4}R^2 + \frac{1}{12}h^2\right)$ and $I_{zz} = \frac{1}{2} m R^2$

Because xyz are principal axes, we can use Euler's Eq.

$$\Sigma \bar{M}_G = \bar{r}_{A/G} \times \left(A_{x'} \bar{i}' + A_{y'} \bar{j}' + A_{z'} \bar{k}'\right) + \bar{r}_{B/G} \times \left(B_{x'} \bar{i}' + B_{y'} \bar{j}'\right) + \Gamma \bar{k}'$$

$$\bar{r}_{A/G} = +\frac{L}{2} \bar{k}' \qquad \bar{r}_{B/G} = -\frac{L}{2} \bar{k}'$$

$$\Sigma \bar{M}_G = \left(\frac{L}{2} A_{x'} \bar{j}' - \frac{L}{2} A_{y'} \bar{i}'\right) + \left(-\frac{L}{2} B_{x'} \bar{j}' + \frac{L}{2} B_{y'} \bar{i}'\right) + \Gamma \bar{k}'$$

$$= \left(\frac{L}{2} B_{y'} - \frac{L}{2} A_{y'}\right) \bar{i}' + \left(\frac{L}{2} A_{x'} - \frac{L}{2} B_{x'}\right) \bar{j}' + \Gamma \bar{k}'$$

$$= \left(\frac{L}{2} B_{y'} - \frac{L}{2} A_{y'}\right)\left(\cos\theta \,\bar{i} - \sin\theta \,\bar{k}\right) + \left(\frac{L}{2} A_{x'} - \frac{L}{2} B_{x'}\right) \bar{j} + \Gamma\left(\sin\theta \,\bar{i} + \cos\theta \,\bar{k}\right)$$ $\leftarrow$ resolve into xyz

$$= \left[\frac{L}{2}\left(B_{y'} - A_{y'}\right)\cos\theta + \Gamma \sin\theta\right] \bar{i} + \left[\frac{L}{2}\left(A_{x'} - B_{x'}\right)\right] \bar{j} + \left[-\frac{L}{2}\left(B_{y'} - A_{y'}\right)\sin\theta + \Gamma \cos\theta\right] \bar{k}$$

Example 6.1 (cont.)

$$\sum \bar{M}_G = \left[\frac{L}{2}(B_y' \cdot A_y')\cos\theta + \Gamma\sin\theta\right]\bar{i} + \left[\frac{L}{2}(A_x' - B_x')\right]\bar{j} + \left[-\frac{L}{2}(B_y' \cdot A_y')\sin\theta + \Gamma\cos\theta\right]\bar{k}$$

$$\sum \bar{M}_A \cdot \bar{i} = I_{xx}\alpha_x - \cancel{(I_{yy} - I_{zz})\omega_y\omega_z}^{\,0}$$

$$\sum \bar{M}_A \cdot \bar{j} = \cancel{I_{yy}\alpha_y}^{\,0} - (I_{zz} - I_{xx})\omega_x\omega_z$$

$$\sum M_A \cdot \bar{k} = I_{zz}\alpha_z - \cancel{(I_{xx} - I_{yy})\omega_x\omega_y}^{\,0}$$

$A = G$ in this case

So

$$\boxed{\begin{aligned} \sum \bar{M}_G \cdot \bar{i} &= \left[\frac{L}{2}(B_y' \cdot A_y')\cos\theta + \Gamma\sin\theta\right] = I_{xx}\alpha_x \\ \sum M_G \cdot \bar{j} &= \left[\frac{L}{2}(A_x' - B_x')\right] = (I_{zz} - I_{xx})\omega_x\omega_z \\ \sum M_G \cdot \bar{k} &= \left[-\frac{L}{2}(B_y' \cdot A_y')\sin\theta + \Gamma\cos\theta\right] = I_{zz}\alpha_z \end{aligned}}$$

Now, formulate the force equations:

$$\sum \bar{F} = (A_x' + B_x')\bar{i}' + (A_y' + B_y')\bar{j}' + (A_z')\bar{k}' = m\bar{a}_G$$

but $\bar{a}_G = 0$, so

$$\boxed{A_x' + B_x' = 0, \quad A_y' + B_y' = 0, \quad \text{and} \quad A_z' = 0}$$

Equations of motion

To find $\Omega(t)$ and the reaction forces, solve this system of equations.

Book also formulates the Eq. of Motion in $x'y'z'$ frame.