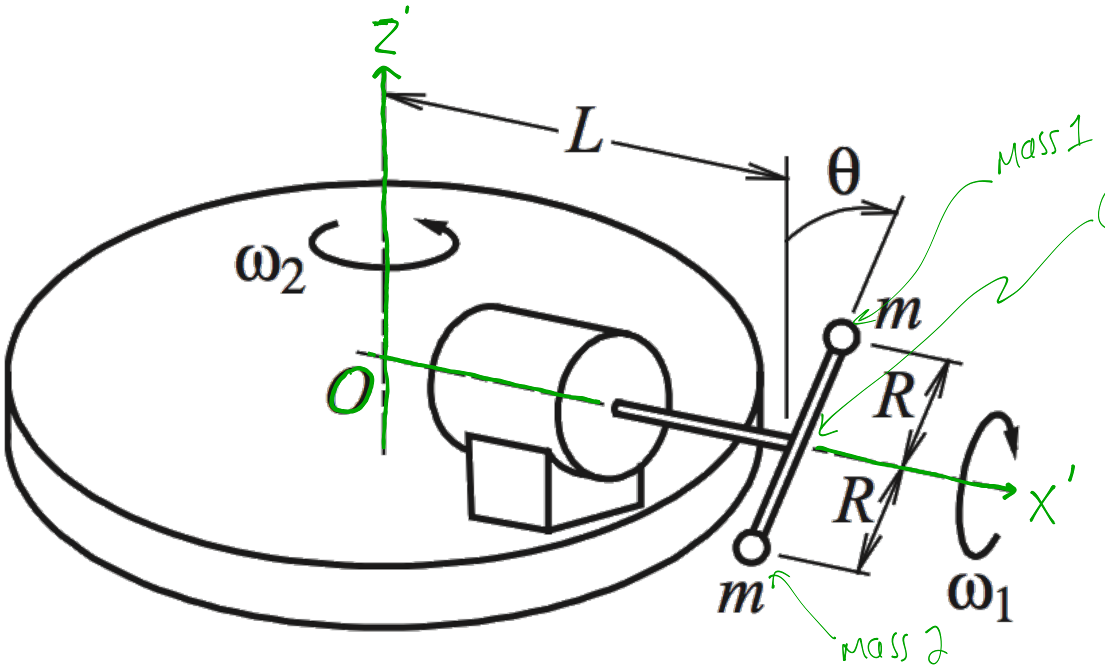
Example 5.1

EXAMPLE 5.1 Identical small spheres having mass m are welded to the ends of a rigid bar that spins about the axis of the motor at angular speed ω_1. The motor is mounted on the horizontal turntable, which rotates at angular speed ω_2. Determine the angular momentum of this pair of particles about point C where the connecting bar is welded to the motor's shaft. Then use the angular momentum to characterize the force system exerted on the connecting bar at point C, as well as the kinetic energy of these spheres. Express the result in terms of the angle θ from the bar's centerline to vertical.



Angular velocity of the two-sphere body is

$$\bar{\omega} = -\omega_1 \bar{i}' + \omega_2 \bar{k}'$$

We can angular momentum about point C:

Q: Why is this a "good" point to evaluate about?

It's the COM of the sphere system.

Define

$$\bar{r}_{1/C} = R\sin\theta\,\bar{j}' + R\cos\theta\,\bar{k}'$$

$$\bar{r}_{2/C} = -R\sin\theta\,\bar{j}' - R\cos\theta\,\bar{k}' = -\bar{r}_{1/C}$$

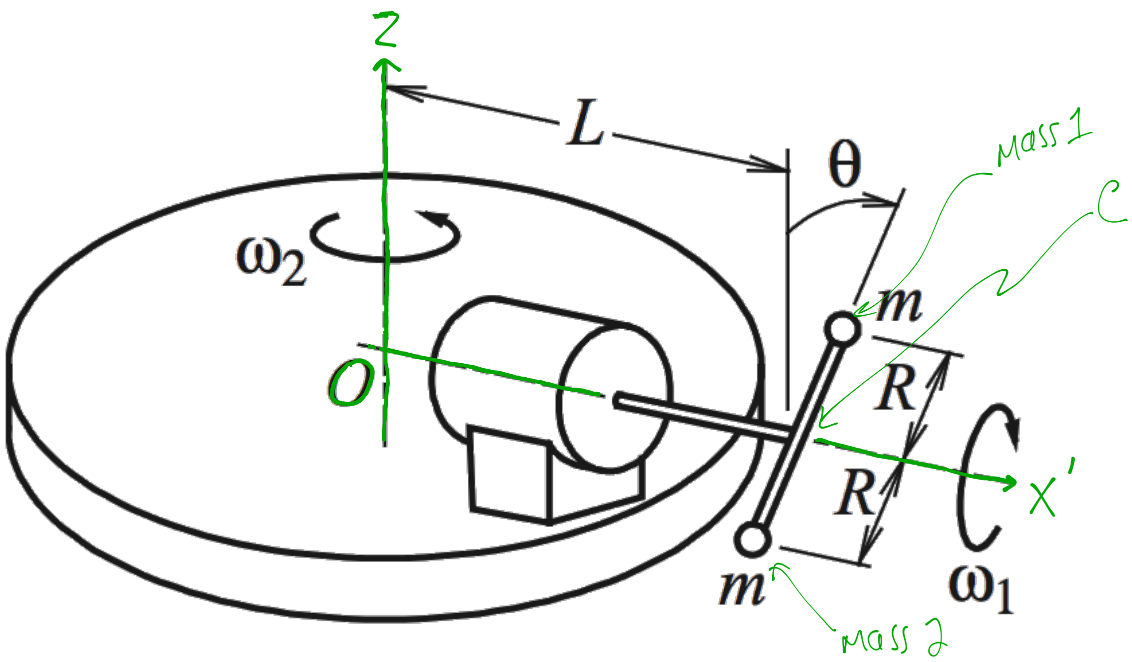
For collections of particles,

$$\bar{H}_A = \sum_{j=1}^{N} m_j \left[\bar{r}_{j/A} \times (\bar{\omega} \times \bar{r}_{j/A})\right] = m\left[\bar{r}_{1/A} \times (\bar{\omega} \times \bar{r}_{1/A})\right] + m\left[\bar{r}_{2/A} \times (\bar{\omega} \times \bar{r}_{2/A})\right]$$

$$\boxed{\bar{H}_C = -2mR^2\omega_1\bar{i}' - 2mR^2\omega_2^2\sin\theta\cos\theta\,\bar{j}' + 2mR^2\omega_2\sin^2\theta\,\bar{k}'}$$

Note:
- not constant even though ω_1 and ω_2 are
- $\bar{i}'$ and $\bar{j}'$ change too

Example 5.1 (cont.)



Now, we want the forces on the bar

If rotating quickly, gravity has only small effect. Ignore it.

Point C is the COM. It follows a circular path:

$$\boxed{\Sigma F = 2m\bar{a}_c = 2m\left(-L\omega_2^2\,\bar{i}'\right)}$$

Can get from looking at path of C

To find the moments, look at moment on shaft at C

$$\boxed{\Sigma \bar{M}_c = \dot{\bar{H}}_c = \frac{\partial \bar{H}_c}{\partial t} + \bar{\omega}_{x'y'z'} \times \bar{H}_c \qquad \omega_{x'y'z'} = \omega_2 k'}$$

angular velocity of i'j'k' frame

To find the kinetic energy use:

$$T = \frac{1}{2} m \bar{v}_G \cdot \bar{v}_G + \frac{1}{2}\bar{\omega} \cdot \bar{H}_G$$

$$= \frac{1}{2}(2m)(L\omega_2)^2 + \frac{1}{2}\left(-\omega_1 \bar{i}' + \omega_2 k'\right) \cdot \bar{H}_c$$

$$\boxed{T = mR^2\omega_1^2 + m\omega_2^2\left(L^2 + R^2\sin^2\theta\right)}$$

Q: How else could we find this?
Use total velocity of each particle

Rate of Change of Angular Momentum (Sec. 5.3)

We have not yet defined how xyz rotates. In addition, the body angular velocity will likely be different than the frame's, meaning inertia properties change too.

To avoid these complications, make xyz a body-fixed coord. system.

This means that $\bar{\Omega} = \bar{\omega}$

ang. vel of frame

ang. vel of body

So

$$\bar{\alpha} = \frac{d\bar{\omega}}{dt} = \frac{\partial \bar{\omega}}{\partial t} + \cancelto{0}{\bar{\Omega} \times \bar{\omega}} = \dot{\omega}_x \bar{i} + \dot{\omega}_y \bar{j} + \dot{\omega}_z \bar{k}$$

$$\bar{\alpha} = \frac{\partial \bar{\omega}}{\partial t} \rightarrow \dot{\omega}_p = \alpha_p \text{ for } p = i, j, k \quad \text{(if xyz is body-fixed)}$$

leads to this

And

$$\dot{\bar{H}}_A = \frac{\partial \bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A$$

$$\frac{\partial H_A}{\partial t} = \left(I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z\right)\bar{i} + \left(I_{yy}\alpha_y - I_{yx}\alpha_x - I_{yz}\alpha_z\right)\bar{j} + \left(I_{zz}\alpha_z - I_{zx}\alpha_x - I_{zy}\alpha_y\right)\bar{k}$$

Using the matrix notation:

$$\dot{\bar{H}}_A = I\bar{\alpha} + \bar{\omega} \otimes I\bar{\omega}$$

matrix implementation of a cross product

If we choose the axes as principal axes, then:

$$\left.\begin{aligned} \dot{H}_A = &\left[I_{xx}\alpha_x - \left(I_{yy} - I_{zz}\right)\omega_y\omega_z\right]\bar{i} \\ + &\left[I_{yy}\alpha_y - \left(I_{zz} - I_{xx}\right)\omega_x\omega_z\right]\bar{j} \\ + &\left[I_{zz}\alpha_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y\right]\bar{k} \end{aligned}\right\} \text{ Euler's Equations}$$

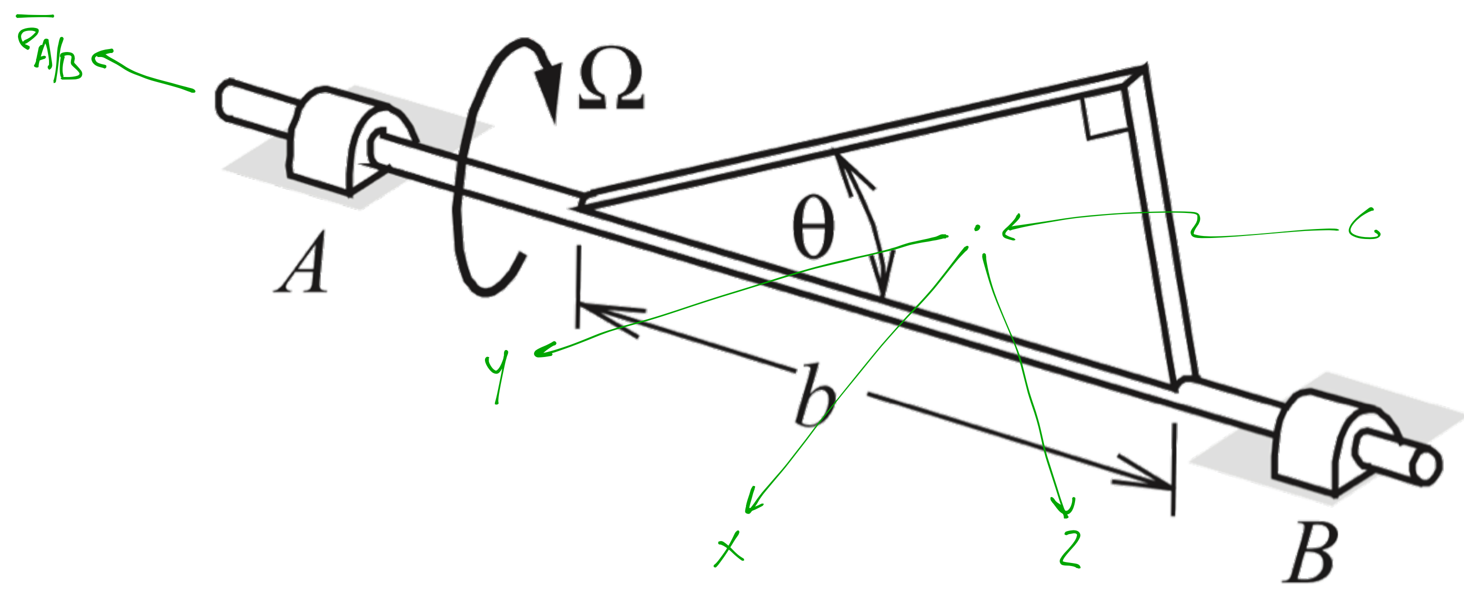
Q: Why would we not want to use principal axes?

Can make finding $\bar{\omega}$ and $\bar{\alpha}$ difficult

Example 5.10

EXAMPLE 5.10 The right triangular plate is welded along its hypotenuse to a shaft that rotates at the variable rate Ω. Determine $d\bar{H}_G/dt$ for the plate. For the special case in which Ω is constant, predict which way the dynamic reactions generated at the bearings will be oriented.

xyz fixed to body at COM



Can look up inertia properties in table. Axes should match common description (and exactly match those in the book)

Find that

$$I_{xx} = \frac{1}{18} mb^2 \quad , \quad I_{yy} = \frac{1}{18} mb^2 \sin^2\theta \quad , \quad I_{zz} = \frac{1}{18} mb^2 \cos^2\theta$$

$$I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0 \qquad I_{yz} = I_{zy} = \frac{-1}{36} mb^2 \sin\theta\cos\theta$$

Angular velocity is:

$$\bar{\omega} = \Omega \bar{e}_{A/B} \qquad \bar{e}_{A/B} = \cos\theta \bar{J} - \sin\theta \bar{K}$$

$$\bar{\omega} = \Omega(\cos\theta \bar{J} - \sin\theta \bar{K})$$

Angular acceleration is

$$\bar{\alpha} = \dot{\bar{\omega}} = \frac{\partial \bar{\omega}}{dt} = \dot{\Omega} \bar{e}_{A/B} + \Omega \cancel{\dot{\bar{e}}_{A/B}}^{\,0} = \dot{\Omega} \bar{e}_{A/B}$$

$$\bar{\alpha} = \dot{\Omega}(\cos\theta \bar{J} - \sin\theta \bar{K})$$

Example 5.10 (cont.)

$$\bar{\omega} = \Omega(\cos\theta\,\bar{j} - \sin\theta\,\bar{k}) \qquad \bar{\alpha} = \dot{\Omega}(\cos\theta\,\bar{j} - \sin\theta\,\bar{k})$$

$$I_{xx} = \frac{1}{18}mb^2 \,,\quad I_{yy} = \frac{1}{18}mb^2\sin^2\theta \,,\quad I_{zz} = \frac{1}{18}mb^2\cos^2\theta$$

$$I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0 \qquad I_{yz} = I_{zy} = \frac{-1}{36}mb^2\sin\theta\cos\theta$$

We know that Angular Momentum about G is then:

$$\bar{H}_G = (I_{xx}\omega_x - \cancel{I_{xy}}\omega_y - \cancel{I_{xz}}\omega_z)\bar{i} + (I_{yy}\omega_y - \cancel{I_{yx}}\omega_x - I_{yz}\omega_z)\bar{j} + (I_{zz}\omega_z - \cancel{I_{zx}}\omega_x - I_{zy}\omega_y)k$$

$$\bar{H}_G = I_{xx}\omega_x\,\bar{i} + (I_{yy}\omega_y - I_{yz}\omega_z)\bar{j} + (I_{zz}\omega_z - I_{zy}\omega_y)\bar{k}$$

Q: What else can we eliminate in this equation?

$$\bar{\omega} = \Omega(\cos\theta\,\bar{j} - \sin\theta\,\bar{k}) = \Omega\cos\theta\,\bar{j} - \Omega\sin\theta\,\bar{k} = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}$$

$$\omega_x = 0 \,,\quad \omega_y = \Omega\cos\theta \,,\quad \omega_z = -\Omega\sin\theta$$

$$\bar{H}_G = \cancel{I_{xx}\omega_x}\,\bar{i} + (I_{yy}\omega_y - I_{yz}\omega_z)\bar{j} + (I_{zz}\omega_z - I_{zy}\omega_y)\bar{k}$$

← just fill in terms + simplify

The rate of change of this vector is

$$\dot{\bar{H}}_G = \frac{d\bar{H}_G}{dt} = \frac{\partial \bar{H}_G}{\partial t} + \bar{\omega}\times\bar{H}_G$$

We know $\frac{\partial H_G}{\partial t}$ is

$$\frac{\partial H_G}{\partial t} = (\cancel{I_{xx}\alpha_x} - \cancel{I_{xy}}\alpha_y - \cancel{I_{xz}}\alpha_z)\bar{i} + (I_{yy}\alpha_y - \cancel{I_{yx}}\alpha_x - I_{yz}\alpha_z)\bar{j} + (I_{zz}\alpha_z - \cancel{I_{zx}}\alpha_x - I_{zy}\alpha_y)\bar{k}$$

Q: What is $\bar{\omega}$ in that derivative?

Angular vel of the frame that the vector is described in ... here, we have body-fixed coords., so that $\bar{\omega}$ matches the body's $\bar{\omega}$

From here, just plug in terms and simplify.