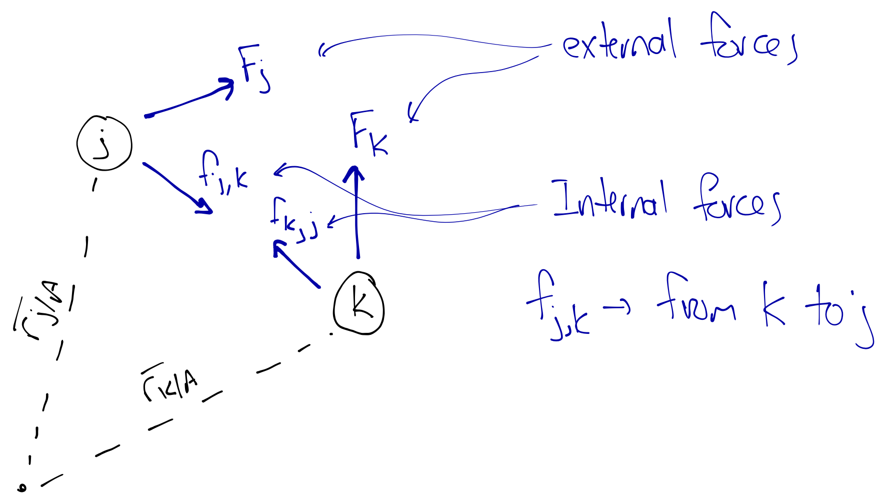
Chapter 5 - Inertial Effects for a Rigid Body

Linear and Angular Momentum (Sec. 5.1)



We know (from Newton) that:

$$\bar{f}_{k,j} = -\bar{f}_{j,k} \longleftarrow \text{remember that these are vectors}$$

Because of this, the moments that the internal forces create are also "equal but opposite"

$$\bar{r}_{k/A} \times \bar{f}_{k,j} = -\bar{r}_{j/A} \times \bar{f}_{j,k}$$

Newton's 2nd Law for each particle

$$\Sigma \bar{F}_j = m_j \bar{a}_j \longrightarrow \bar{F}_j + \bar{f}_{j,k} = m_j \bar{a}_j$$

$$\Sigma \bar{F}_K = m_K \bar{a}_K \longrightarrow \bar{F}_K + \bar{f}_{k,j} = m_K \bar{a}_K$$

Sum all forces on the particle system:

$$\Sigma \bar{F} = \Sigma \bar{F}_j + \Sigma \bar{F}_K = \bar{F}_j + \bar{F}_K = m_j \bar{a}_j + m_K \bar{a}_K \quad \Big\} \text{ only external forces contribute}$$

Similarly, look at the total moment of all forces about A:

$$\Sigma \bar{M}_A = \bar{r}_{j/A} \times \left(\bar{F}_j + \bar{f}_{j,k}\right) + \bar{r}_{k/A} \times \left(\bar{F}_K + \bar{f}_{k,j}\right) = \bar{r}_{j/A} \times m_j \bar{a}_j + \bar{r}_{k/A} \times m_K \bar{a}_K$$

$$= \bar{r}_{j/A} \times \bar{F}_j + \bar{r}_{k/A} \times \bar{F}_K = \bar{r}_{j/A} \times m_j \bar{a}_j + \bar{r}_{k/A} \times m_K \bar{a}_K \quad \Big\} \text{ only external forces contribute}$$

This continues for N-masses.

$$\Sigma \bar{F} = \sum_{j=1}^{N} \bar{F}_j = \sum_{j=1}^{N} m_j \bar{a}_j \quad \text{and} \quad \Sigma \bar{M}_A = \sum_{j=1}^{N} \bar{r}_{j/A} \times \bar{F}_j = \sum_{j=1}^{N} \bar{r}_{j/A} \times m_j \bar{a}_j$$

Linear and Angular Momentum (cont.)

$$\Sigma \bar{F} = \sum_{j=1}^{N} \bar{F}_j = \sum_{j=1}^{N} m_j \bar{a}_j$$

$$= \sum_{j=1}^{N} m_j \left[\frac{d^2}{dt^2} \bar{r}_{j/0} \right] = \frac{d^2}{dt^2} \underbrace{\left[\sum_{j=1}^{N} m_j \bar{r}_{j/0} \right]}_{\text{First moment of mass}}$$

Summing mass over the entire system, and defining G as the system COM

$$m_{sys} = \sum_{j=1}^{N} m_j \qquad \bar{r}_{G/0} = X_G \bar{I} + Y_G \bar{J} + Z_G \bar{K}$$

Then

$$m_{sys} \bar{r}_{G/0} = \sum_{j=1}^{N} m_j \left(X_j \bar{I} + Y_j \bar{J} + Z_j \bar{K} \right) = \sum_{j=1}^{N} m_j \bar{r}_{j/0}$$ ← so sub $m_{sys} \bar{r}_{G/0}$ into ΣF eq.

$$\Sigma \bar{F} = \frac{d^2}{dt^2} \left(m_{sys} \bar{r}_{G/0} \right)$$

If the system mass (m_{sys}) is constant then:

$$\Sigma \bar{F} = m_{sys} \bar{a}_G$$ ← Newton predicted motion of rigid body COM

We can follow a similar derivation to find

$$\Sigma \bar{M}_A = m_{sys} \bar{r}_{G/A} \times \bar{a}_A + \frac{d}{dt} \bar{H}_A \quad \text{where} \quad \bar{H}_A = \sum_{j=1}^{N} \left(\bar{r}_{j/A} \times \underbrace{m_j \bar{v}_{j/A}}_{} \right)$$

Angular Momentum

linear momentum of particle j relative to frame whose origin is A. On a rigid body, we can often simplify this!

Rigid Body - Basic Equations (Sec. 5.1.2)

Now, require A to be fixed relative to the body

So,

$$\bar{v}_{j/A} = \bar{\omega} \times \bar{r}_{j/A}$$

And the angular momentum becomes

$$\bar{H}_A = \sum_{j=1}^{N} m_j \left[\bar{r}_{j/A} \times \left(\bar{\omega} \times \bar{r}_{j/A} \right) \right]$$

Looking back to $\Sigma \bar{M}_A$

$$\Sigma \bar{M}_A = \underbrace{M_{sys} \bar{r}_{G/A} \times \bar{a}_A}_{} + \frac{d}{dt} \bar{H}_A$$

This term can cause "trouble", because the analysis of rotation now requires simultaneous info about the motion of the point

Q: To avoid this, what can we do?

Remember that we choose where to sum moments about. So, pick A such that

① A is the center of mass, G

$\bar{r}_{G/A} = 0 \rightarrow$ so $\bar{r}_{G/A} \times \bar{a}_A = 0$ and that term can be ignored

or

② $\bar{a}_A = 0 \leftarrow$ "pure" rotation

or

③ $\bar{r}_{G/A} \times \bar{a}_A = 0 \leftarrow$ acceleration directly toward or away from COM

We will extensively use ① and ②, but almost never use ③

If point A meets condition ①, ②, or ③, then the moment equation reduces to

$$\Sigma \bar{M}_A = \frac{d}{dt} \bar{H}_A$$

Similar form exists for linear motion

$$\Sigma \bar{F} = \frac{d}{dt} \underbrace{\left(M_{sys} \bar{V}_G \right)}_{= P \text{ - linear momentum}}$$

Kinetic Energy (Sec. 5.1.3)

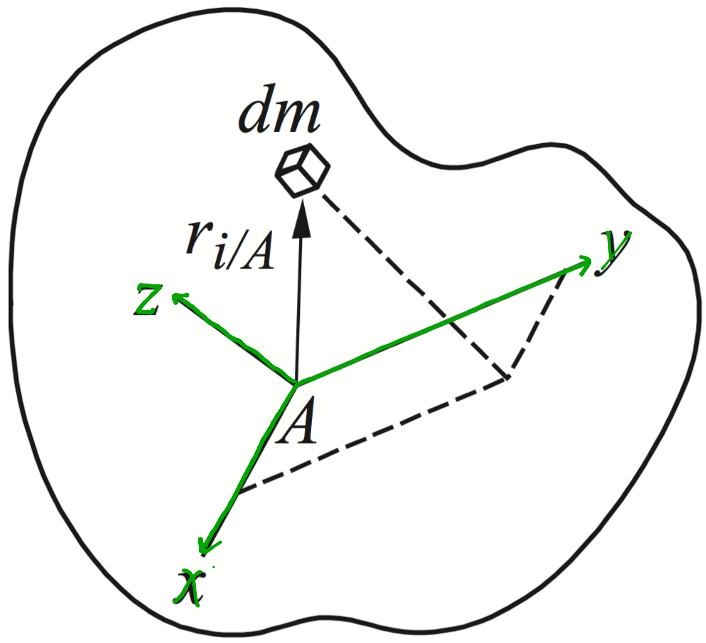
See Book for derivation from collection of points

$$T = \frac{1}{2} M \bar{v}_G \cdot \bar{v}_G + \frac{1}{2} \bar{\omega} \cdot \bar{H}_G$$ ← valid for all motion

$$T = \frac{1}{2} \bar{\omega} \cdot H_O$$ ← only valid for pure rotation about O

Inertia Properties (Sec. 5.2)

Moments and Products of Inertia (Sec. 5.2.1)



for each piece of mass, dm_i, we can write

$$\bar{r}_{i/A} = x\bar{i} + y\bar{j} + z\bar{k} \quad \text{and} \quad \bar{\omega}_i = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}$$

If sum over entire body, summation → integration for ∞ many dm_i, then sub into angular momentum eq:

$$\bar{H}_A = \iiint (x\bar{i} + y\bar{j} + z\bar{k}) \times \left[(\omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}) \times (x\bar{i} + y\bar{j} + z\bar{k})\right] dm$$

This reduces to

$$\bar{H}_A = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\bar{i} + (I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z)\bar{j} + (I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)\bar{k}$$

where

$$I_{xx} = \iiint (y^2 + z^2)\,dm, \quad I_{yy} = \iiint (x^2 + z^2)\,dm, \quad I_{zz} = \iiint (x^2 + y^2)\,dm \leftarrow \text{Moments of Inertia}$$

$$I_{xy} = I_{yx} = \iiint xy\,dm, \quad I_{xz} = I_{zx} = \iiint xz\,dm, \quad I_{yz} = I_{zy} = \iiint yz\,dm \leftarrow \text{Products of Inertia}$$

reflect symmetry (or not) of mass around axes considered

Using these, we can write the angular momentum in matrix form

$$\bar{H}_A = I\omega, \text{ where } \quad I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

If no products of inertia exist for a set of axes, then the axes are principal axes

Finding/using these can simplify the math in Eq. of motion

Moments and Products of Inertia (cont.)

A common way to define moments of inertia is radius of gyration

$$k_P = \left[\frac{I_{PP}}{M}\right]^{1/2} \longrightarrow I_{PP} = \frac{1}{2}Mk_P^2$$ — what radius would a thin ring need to have to have an equiv. moment of inertia as the body.

Transformations (Sec. 5.2.2)

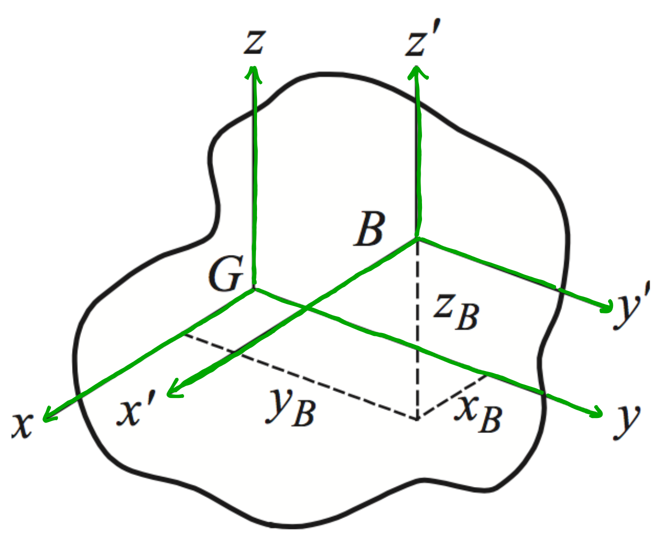
We will not often solve the integrals directly. Many common shapes are known and we can add them to make others.

$$I_{\xi\eta} = \underbrace{(I_{\xi\eta})_1}_{\text{subbody 1}} + \underbrace{(I_{\xi\eta})_2}_{\text{sub-body 2}} + \dots \qquad \text{where } \xi,\eta = x, y, \text{ or } z$$

Total Body = subbody 1 + sub-body 2 + . . .

We often need to "transform" inertia properties between coord. sys. we can use:

Parallel Axis Theorem



$$\bar{r}_{B/G} = x_B\bar{i} + y_B\bar{j} + z_B\bar{k}$$

Find (see book for derivation):

$$I_{x'x'} = I_{xx} + m(y_B^2 + z_B^2)$$

$$I_{y'y'} = I_{yy} + m(x_B^2 + z_B^2)$$

$$I_{z'z'} = I_{zz} + m(x_B^2 + y_B^2)$$

Note: This assumes that xyz is a set of centroidal axes

$$I_{x'y'} = I_{xy} + mx_By_B$$

$$I_{x'z'} = I_{xz} + mx_Bz_B$$

$$I_{y'z'} = I_{yz} + my_Bz_B$$

Rotational Transformation

Transform inertia properties between two sets of axes that share the same origin, but not the same orientation.

Q: How do you think we can do this?

Rotation matrices! (see book for derivation)

$I' = R I R^T$

I' = inertia prop in new coordinate system
I : inertia prop in orig coord system
R = rotation matrix for orig. frame to new frame

We'll skip Sections 5.2.3 and 5.2.4 in lecture. Please do read the and be aware of their content.