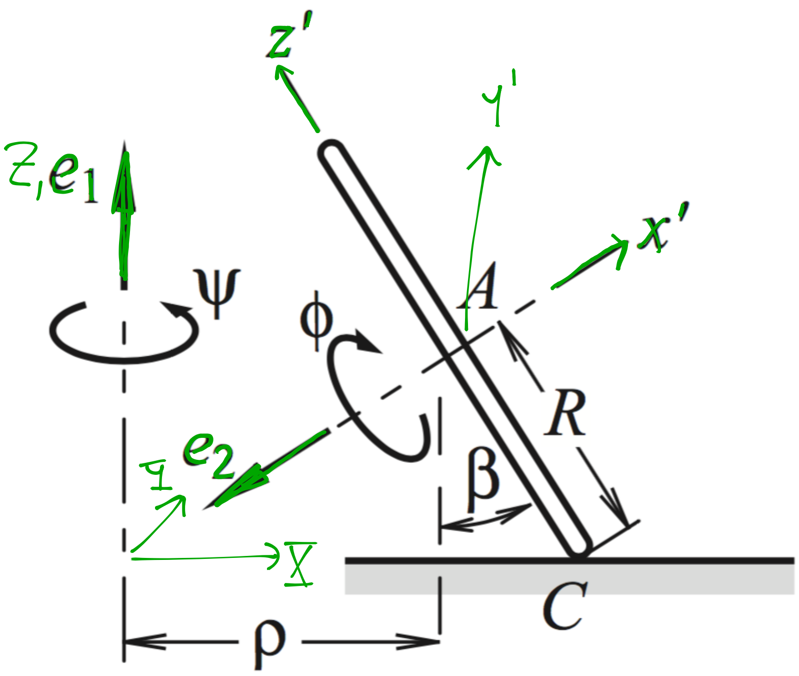
Rolling Disc Example



Assume: β is constant

$\dot{\psi}$ is constant (precession rate)

Follows path of radius ρ

$$\bar{v}_A = \rho\dot{\psi}\,\bar{j}'$$

$$\bar{a}_A = \rho\dot{\psi}\left(\bar{\omega}_{x'y'z'} \times \bar{j}'\right)$$

$$R_\beta = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\omega_{x'y'z'} = \dot{\psi}\,\bar{e}_1 = \dot{\psi}\,\bar{K}$$
$$= \dot{\psi}\left(+\sin\beta\,\bar{i}' + \cos\beta\,\bar{k}'\right)$$

$$\bar{a}_A = \rho\dot{\psi}\left(\dot{\psi}\sin\beta\,\bar{i}' + \dot{\psi}\cos\beta\,\bar{k}' \times \bar{j}\right) = \rho\dot{\psi}\left(\dot{\psi}\sin\beta\,k' - \dot{\psi}\cos\beta\,\bar{i}'\right)$$

$$\bar{\omega} = \dot{\psi}\,\bar{e}_1 + \dot{\phi}\,\bar{e}_2 = \dot{\psi}\,\bar{K} - \dot{\phi}\,\bar{i}' = \dot{\psi}\left(\sin\beta\,\bar{i}' + \cos\beta\,\bar{k}'\right) - \dot{\phi}\,\bar{i}' = \left(\dot{\psi}\sin\beta - \dot{\phi}\right)\bar{i}' + \left(\dot{\psi}\cos\beta\right)\bar{k}'$$

$$\bar{\alpha} = \cancelto{0}{\ddot{\psi}\,\bar{e}_1} + \cancelto{0}{\dot{\psi}\,\dot{\bar{e}}_1} + \cancelto{0}{\ddot{\phi}\,\bar{e}_2} + \dot{\phi}\,\dot{\bar{e}}_2 = \dot{\phi}\left(\bar{\omega} \times \bar{e}_2\right) = \dot{\phi}\left(\bar{\omega} \times -\bar{i}'\right) = -\dot{\phi}\left(\dot{\psi}\cos\beta\right)\bar{j}'$$

If pure rolling, then ground contact point velocity is zero → $\bar{v}_C = 0$

Write $\bar{v}_A$ using this information:

$$\bar{v}_A = \cancelto{0}{\bar{v}_C} + \bar{\omega} \times \bar{r}_{A/C} = \bar{\omega} \times R\bar{k}' = R\left(\dot{\psi}\sin\beta - \dot{\phi}\right)\bar{j}'$$

Equate to earlier expression of $\bar{v}_A$

$$\bar{v}_A = \rho\dot{\psi}\,\bar{j}' = R\left(\dot{\psi}\sin\beta - \dot{\phi}\right)\bar{j}' \longrightarrow \dot{\phi} = \left(\sin\beta - \frac{\rho}{R}\right)\dot{\psi}$$

← spin rate ∝ precession rate

Q: Does this match our intuition?

Can now sub. $\dot{\phi}$ back into $\bar{\omega}$ and $\bar{\alpha}$. ← See that motion can be sufficiently prescribed by Eulerian angles