Chapter 4 - Kinematics of Constrained Rigid Bodies

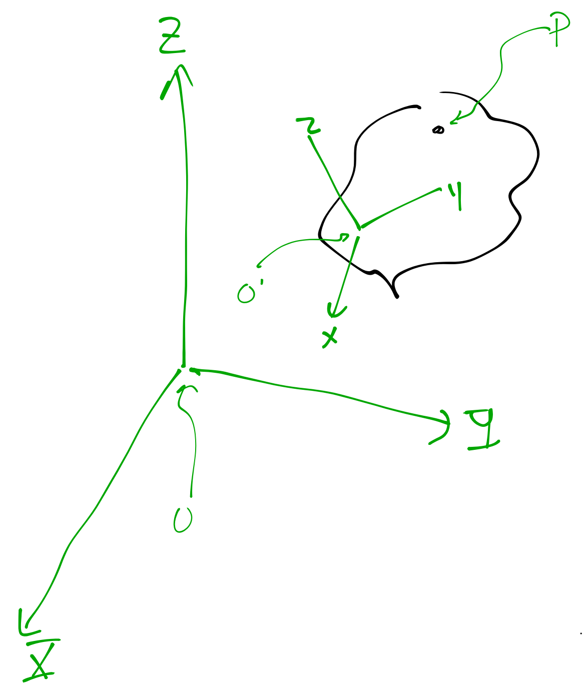
We found previously that

$$\bar{v}_P = \underbrace{\bar{v}_{O'}}_{\text{velocity of the frame's origin}} + \underbrace{(v_P)_{xyz}}_{\text{relative velocity - velocity of P within the frame}} + \underbrace{\bar{\omega} \times \bar{r}_{P/O'}}_{\text{velocity of point due to rotation of the frame, } \bar{\omega}}$$

$$\bar{a}_P = \bar{a}_{O'} + (a_P)_{xyz} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) + 2\bar{\omega} \times (\bar{v}_P)_{xyz}$$

If point P and point O' are on the same rigid body, to which the xyz frame is fixed, then:

$$(v_P)_{xyz} = 0 \qquad \text{and} \qquad (a_P)_{xyz} = 0$$



So

$$\bar{v}_P = \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{P/O'}$$

$$\bar{a}_P = \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'})$$

Chasle's Theorem:

We can write the general motion of a rigid body by the translation of a point (with constant orientation), then a pure rotation about that point.

Key: Which point we choose is arbitrary

Eulerian Angles (Sec. 4.1)

Describe the rotation of a body by a:

1. precession - about a fixed Z axis

2. nutation - about the precessed y-axis, typically written y'

3. spin - about the nutated z-axis, typically written z''

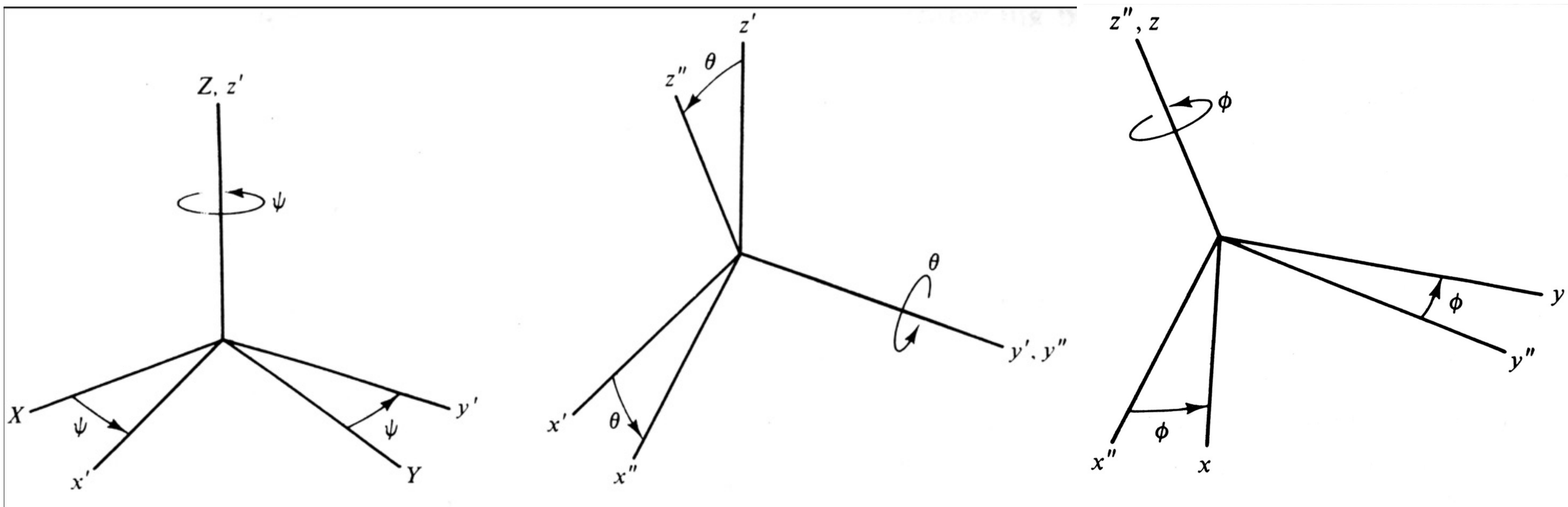


Figure 4.2 Precession. **Figure 4.3** Nutation. **Figure 4.4** Spin.

Let's write the rotation matrices describing these rotations.

Precession - $XYZ \rightarrow x'y'z'$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_\psi \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{where } R_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Nutation - $x'y'z' \rightarrow x''y''z''$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = R_\theta \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad \text{where } R_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Spin - $x''y''z'' \rightarrow xyz$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\phi \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \quad \text{where } R_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eulerian Angles (cont.)

So, the total transformation is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\phi R_\theta R_\psi \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

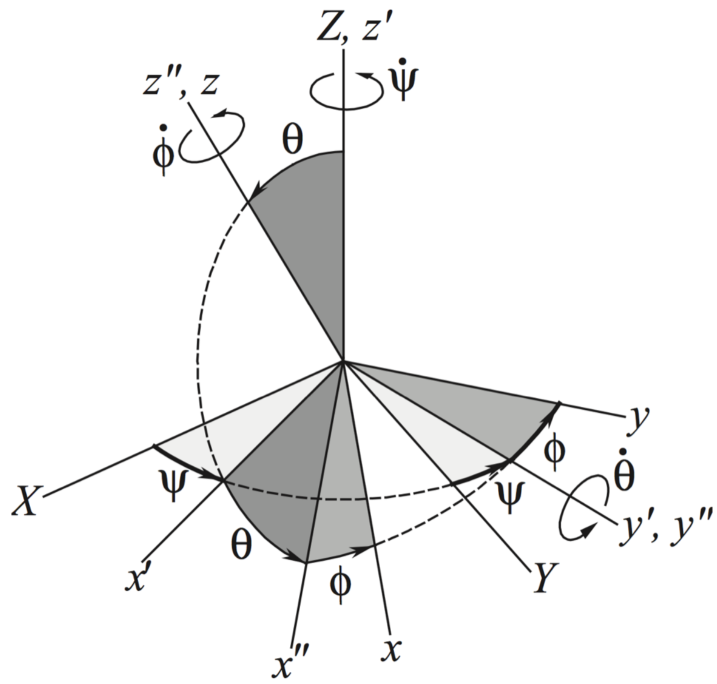


Figure 4.2. Definition of the Eulerian angles.

We can write angular velocity and angular accel. by adding rotation rates about the various axes.

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k}'' = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k} \quad \leftarrow \bar{k}'' = \bar{k}$$

Write these in array notation – $\bar{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

$$\bar{\omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}_{XYZ} + \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}_{x'y'z'} + \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}_{xyz}$$

← Now, use the rotation matrices to resolve into a single coord. system

$$\bar{\omega} = \underbrace{R_\phi R_\theta R_\psi}_{\text{rotation from } XYZ \text{ to } xyz} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}_{XYZ} + \underbrace{R_\phi R_\theta}_{\text{rotation from } x'y'z' \text{ to } xyz} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}_{x'y'z'} + \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}_{xyz}$$

$$= \begin{bmatrix} -\dot{\psi}\sin\theta\cos\phi \\ \dot{\psi}\sin\theta\sin\phi \\ \dot{\psi}\cos\theta \end{bmatrix}_{xyz} + \begin{bmatrix} \dot{\theta}\sin\phi \\ \dot{\theta}\cos\phi \\ 0 \end{bmatrix}_{xyz} + \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}_{xyz} = \begin{bmatrix} -\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi \\ \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi \\ \dot{\psi}\cos\theta + \dot{\phi} \end{bmatrix}_{xyz} \begin{matrix} \leftarrow \omega_x \\ \leftarrow \omega_y \\ \leftarrow \omega_z \end{matrix} \Bigg\} \; \bar{\omega} = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}$$

This is valid for <u>all</u> time, so $\bar{\alpha} = \frac{d\bar{\omega}}{dt}$

$$\frac{d}{dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \bar{\omega}\times(\cdot) \rightarrow \bar{\alpha} = \left[\dot{\omega}_x\bar{i} + \dot{\omega}_y\bar{j} + \dot{\omega}_z\bar{k}\right] + \left[\cancel{\bar{\omega}\times\bar{\omega}}^{\,0}\right]$$