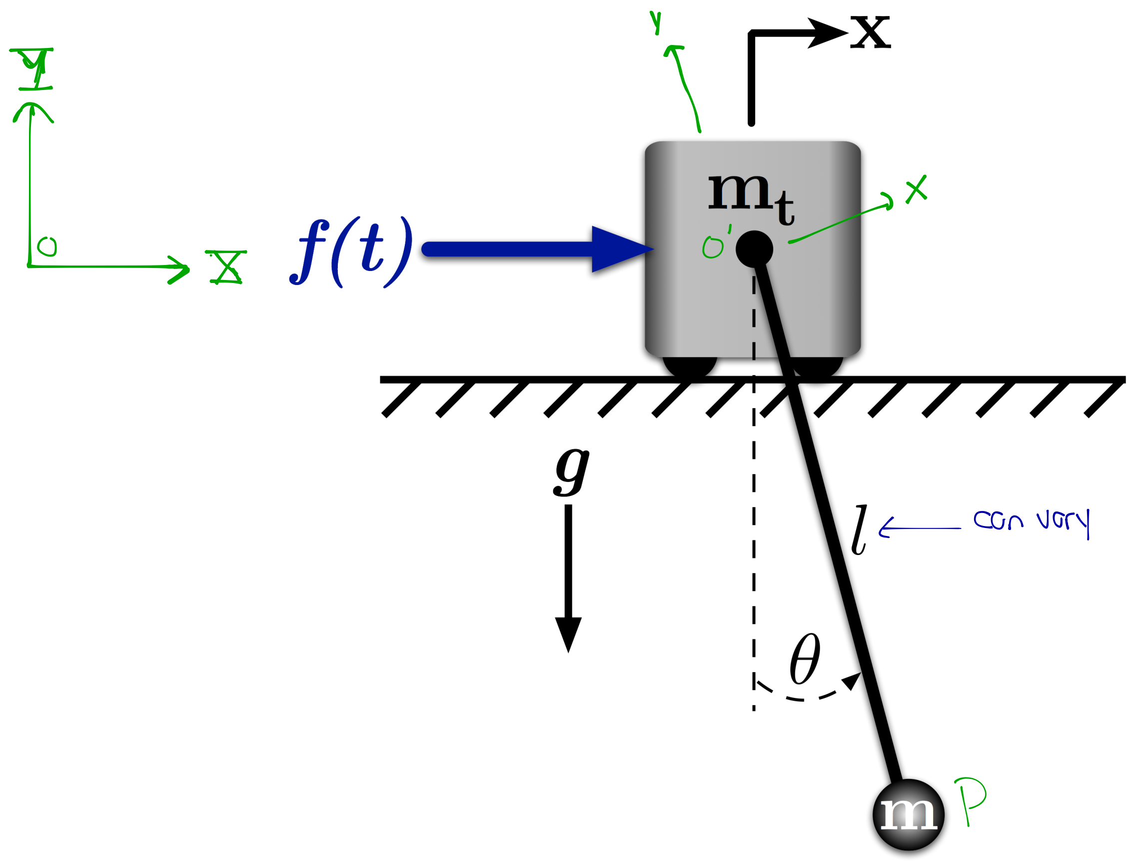
A Simple (Planar) Example

Write the velocity and acceleration of the crane payload in terms of the fixed frame XYZ.



For this simple system we could easily solve $\frac{d\bar{r}_{P/O}}{dt}$ direct. But, let's use the relationships we just learned.

$$\bar{v}_P = \bar{v}_{O'} + (v_P)_{xyz} + \bar{\omega}\times\bar{r}_{P/O'}$$

Q: What is $(v_P)_{xyz}$?

$$(v_P)_{xyz} = -\dot{l}\,\bar{j}$$

Q: $\bar{v}_{O'}$? $\longrightarrow$ $\dot{x}\,\bar{I}$

Q: $\bar{\omega}\times\bar{r}_{P/O'}$?

$$\bar{\omega} = \dot{\theta}\bar{K} = \dot{\theta}\bar{k} \qquad \bar{r}_{P/O'} = -l\,\bar{j}$$

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} I \\ J \\ K \end{bmatrix}$$

$$\bar{\omega}\times\bar{r}_{P/O'} = \dot{\theta}\bar{k}\times(-l)\bar{j} = l\dot{\theta}\,\bar{i}$$

So, $\bar{v}_P = \dot{x}\bar{I} - \dot{l}\bar{j} + l\dot{\theta}\bar{i} = \dot{x}\bar{I} - \dot{l}(-\sin\theta\,\bar{I} + \cos\theta\,\bar{J}) + l\dot{\theta}(\cos\theta\,\bar{I} + \sin\theta\,\bar{J})$

$$\boxed{\bar{v}_P = (\dot{x} + \dot{l}\sin\theta + l\dot{\theta}\cos\theta)\bar{I} + (-\dot{l}\cos\theta + l\dot{\theta}\sin\theta)\bar{J}}$$

$$\bar{a}_P = \bar{a}_{O'} + (a_P)_{xyz} + \bar{\alpha}\times\bar{r}_{P/O'} + \bar{\omega}\times(\bar{\omega}\times\bar{r}_{P/O'}) + 2\bar{\omega}\times(v_P)_{xyz}$$

$$\bar{a}_{O'} = \ddot{x}\bar{I} \qquad (a_P)_{xyz} = -\ddot{l}\bar{j} \qquad \bar{\alpha} = \ddot{\theta}\bar{K} = \ddot{\theta}\bar{k}$$

already found that $(\omega\times\bar{r}_{P/O'}) = l\dot{\theta}\bar{i}$ $\longrightarrow$ $\bar{\omega}\times(\bar{\omega}\times\bar{r}_{P/O'}) = \dot{\theta}\bar{k}\times(l\dot{\theta}\bar{i}) = l\dot{\theta}^2\bar{j}$

$$\bar{\alpha}\times\bar{r}_{P/O'} = \ddot{\theta}\bar{k}\times -l\bar{j} = l\ddot{\theta}\bar{i}$$

$$2\bar{\omega}\times(v_P)_{xyz} = 2\dot{\theta}\bar{k}\times -\dot{l}\bar{j} = 2\dot{l}\dot{\theta}\bar{i}$$

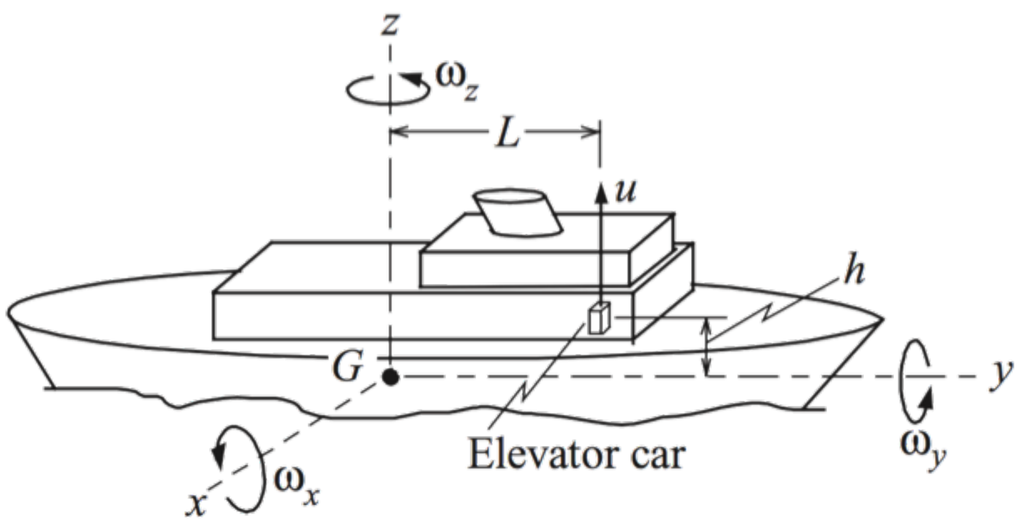
$$\bar{a}_P = \ddot{x}\bar{I} - \ddot{l}\bar{j} + l\ddot{\theta}\bar{i} + l\dot{\theta}^2\bar{j} + 2\dot{l}\dot{\theta}\bar{i} = \ddot{x}\bar{I} + (l\ddot{\theta} + 2\dot{l}\dot{\theta})\bar{i} + (l\dot{\theta}^2 - \ddot{l})\bar{j}$$

$$= \ddot{x}\bar{I} + (l\ddot{\theta} + 2\dot{l}\dot{\theta})(\cos\theta\,\bar{I} + \sin\theta\,\bar{J}) + (l\dot{\theta}^2 - \ddot{l})(-\sin\theta\,\bar{I} + \cos\theta\,\bar{J})$$

$$\boxed{\bar{a}_P = \left[\ddot{x} + (l\ddot{\theta} + 2\dot{l}\dot{\theta})\cos\theta - (l\dot{\theta}^2 - \ddot{l})\sin\theta\right]\bar{I} + \left[(l\ddot{\theta} + 2\dot{l}\dot{\theta})\sin\theta + (l\dot{\theta}^2 - \ddot{l})\cos\theta\right]\bar{J}}$$

Example 3.13

EXAMPLE 3.13 Let ω_x, ω_y, and ω_z denote the pitch, roll, and yaw rates, respectively, of a ship about xyz axes that are attached to the ship with the orientations shown. All of these rotation rates are variable quantities. The origin of xyz coincides with the center of mass G of the ship. Consider an elevator car whose path perpendicularly intersects the centerline at a distance L forward from the center of mass. Let $h(t)$ denote the height of the car above the centerline. The velocity and acceleration of the center of mass at this instant are $\bar{v}_G$ and $\bar{a}_G$. Determine the corresponding velocity and acceleration of the car.



Example 3.13

$$\bar\omega = \omega_x \bar i + \omega_y \bar j + \omega_z \bar k$$

$$\bar\alpha = \dot\omega_x \bar i + \dot\omega_y \bar j + \dot\omega_z \bar k + \omega_x(\bar\omega\times\bar i) + \omega_y(\bar\omega\times\bar j) + \omega_z(\bar\omega\times\bar k)$$

$$= \dot\omega_x \bar i + \dot\omega_y \bar j + \dot\omega_z \bar k + \underbrace{\bar\omega\times(\omega_x\bar i+\omega_y\bar j+\omega_z\bar k)}_{\bar\omega\times\bar\omega=0} = \dot\omega_x \bar i + \dot\omega_y \bar j + \dot\omega_z \bar k$$

← We'll see later that this is always true for angular accel in body-fixed coordinates.

Position of the elevator relative to frame origin:

$$\bar r_{E/G} = L\bar j + h\bar k$$

$$(v_E)_{xyz} = \dot h\bar k \quad \text{and} \quad (a_E)_{xyz} = \ddot h\bar k$$

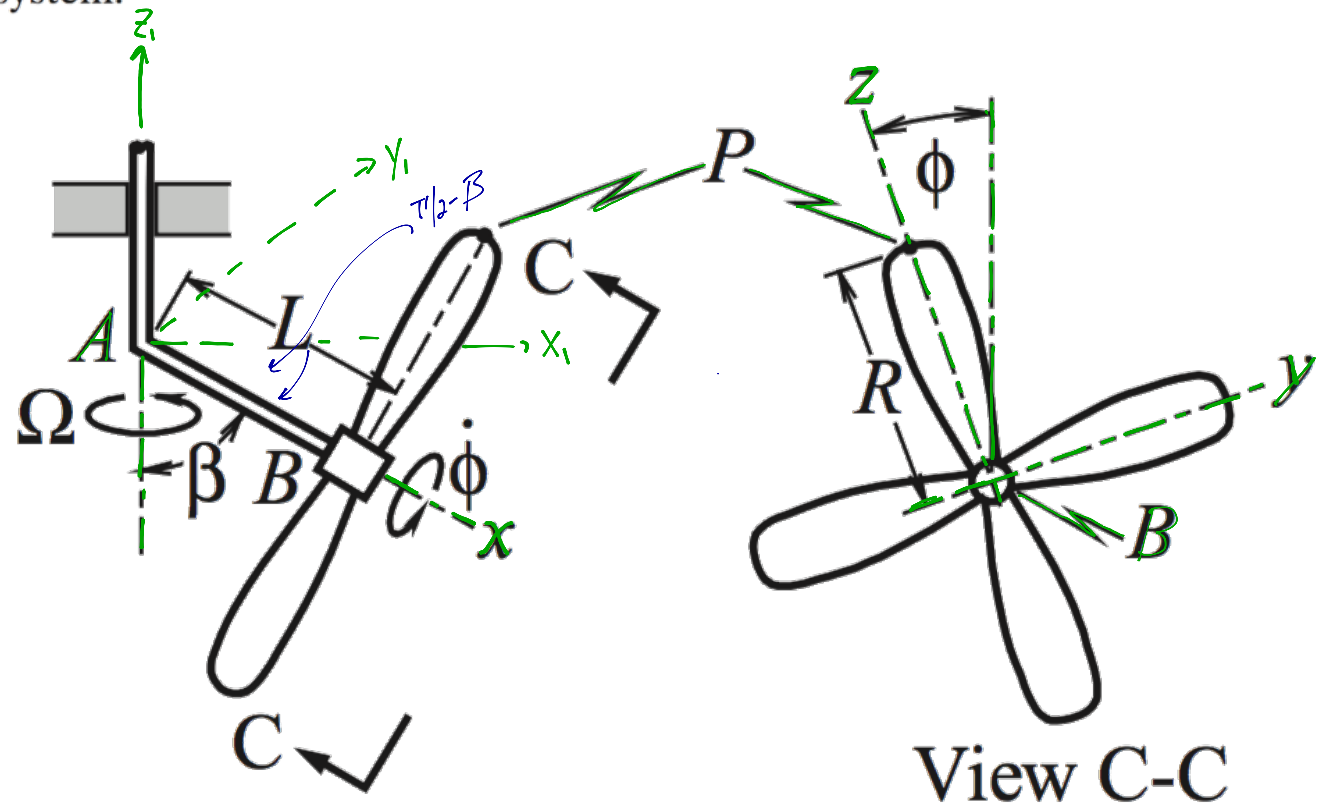
So, $$\bar v_P = \bar v_G + (v_E)_{xyz} + \bar\omega\times\bar r_{E/G} = \bar v_G + \dot h\bar k + (\omega_x\bar i+\omega_y\bar j+\omega_z\bar k)\times(L\bar j+h\bar k)$$

$$\boxed{\bar v_P = \bar v_G + (h\omega_y - L\omega_z)\bar i - \omega_x h\bar j + (\dot h + L\omega_x)\bar k}$$

$$\bar a_P = \bar a_G + (a_E)_{xyz} + \bar\alpha\times\bar r_{E/G} + \bar\omega\times(\bar\omega\times\bar r_{E/G}) + 2\bar\omega\times(v_E)_{xyz}$$ ← just plug in values and solve. See book for solution

Example 3.14

EXAMPLE 3.14 The cooling fan consists of a shaft that rotates about the vertical axis at angular speed Ω as the blades rotate around the shaft at angular rate $\dot{\phi}$, where ϕ is the angle of rotation of one of the blades from the top-center position. Both rotation rates are constant. Derive expressions for the velocity and acceleration of the blade tip P in terms of components relative to the body-fixed xyz reference system.



xyz - fixed to blade

$x_1y_1z_1$ - fixed to vertical bar, z_1 "up"

Angular velocities are then:

$$\bar{\omega} = \Omega \bar{k}_1 + \dot{\phi}\,\bar{i} \quad \leftarrow \text{angular velocity of fan}$$

$$\bar{\Omega}_1 = \Omega \bar{k}_1 \quad \leftarrow \text{angular velocity of } x_1y_1z_1 \text{ frame}$$

$$\Omega_2 = \bar{\omega} \quad \leftarrow \text{angular velocity of } xyz \text{ frame (fixed to the fan so = fan ang. vel.)}$$

Angular accelerations: rotation rates are constant

$$\bar{\alpha} = \frac{d\bar{\omega}}{dt} = \cancel{\dot{\Omega}}^{\,0}\bar{k}_1 + \Omega\dot{\bar{k}}_1 + \cancel{\ddot{\phi}}^{\,0}\bar{i} + \dot{\phi}\dot{\bar{i}}$$

$$= \Omega\left(\cancel{\bar{\Omega}_1 \times \bar{k}_1}\right)^0 + \dot{\phi}\left(\bar{\omega}\times\bar{i}\right) = \dot{\phi}\left(\bar{\omega}\times\bar{i}\right)$$

Q: We need to write $\bar{k}_1$ in the xyz frame. How?

2 rotations, ① align $x'y'z'$ with x' along AB link ($\bar{i}' = \bar{i}$) → rotation of $\pi/2-\beta$ about $\bar{j}'$

② rotate by ϕ along x axis ⟶ ϕ about $\bar{i}$

Write ① as R_y and ② as R_x

Example 3.14 (cont.)

They are body fixed rotations, so the total Rotation matrix, R, is

$$R = R_x R_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos(\pi/2-\beta) & 0 & -\sin(\pi/2-\beta) \\ 0 & 1 & 0 \\ \sin(\pi/2-\beta) & 0 & \cos(\pi/2-\beta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \sin\beta & 0 & -\cos\beta \\ 0 & 1 & 0 \\ \cos\beta & 0 & \sin\beta \end{bmatrix}$$

$$R = \begin{bmatrix} \sin\beta & 0 & -\cos\beta \\ \sin\phi\cos\beta & \cos\phi & \sin\phi\sin\beta \\ \cos\phi\cos\beta & -\sin\phi & \cos\phi\sin\beta \end{bmatrix} \quad \text{so}$$

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = R \begin{bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{bmatrix} = R^T \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} \longrightarrow \bar{k}_1 = -\cos\beta\,\bar{i} + \sin\phi\sin\beta\,\bar{j} + \cos\phi\sin\beta\,\bar{k}$$

$$\bar{\omega} = \Omega\bar{k}_1 + \dot{\phi}\bar{i} = \Omega\left(-\cos\beta\,\bar{i} + \sin\phi\sin\beta\,\bar{j} + \cos\phi\sin\beta\,\bar{k}\right) + \dot{\phi}\bar{i}$$

$$= \left(\dot{\phi} - \Omega\cos\beta\right)\bar{i} + \Omega\sin\phi\sin\beta\,\bar{j} + \cos\phi\sin\beta\,\bar{k}$$

$$\bar{\alpha} = \dot{\phi}\left(\bar{\omega}\times\bar{i}\right) = \dot{\phi}\left[\left(\left(\dot{\phi} - \Omega\cos\beta\right)\bar{i} + \Omega\sin\phi\sin\beta\,\bar{j} + \cos\phi\sin\beta\,\bar{k}\right)\times\bar{i}\right]$$

$$= \Omega\dot{\phi}\left(\cos\phi\sin\beta\,\bar{j} - \sin\phi\sin\beta\,\bar{k}\right)$$

We can now start writing expressions for velocity and accel of point p

$$\bar{v}_p = \bar{v}_B + \bar{\omega}\times\bar{r}_{P/B} \quad \text{and} \quad \bar{a}_p = \bar{a}_B + \bar{\alpha}\times\bar{r}_{P/B} + \bar{\omega}\times\left(\bar{\omega}\times\bar{r}_{P/B}\right) \leftarrow \text{work from point B}$$

collect terms in xyz and simplify

Q: v_B and a_B = ?

$$\bar{v}_B = \cancel{v_A} + \bar{\Omega}_1\times\bar{r}_{B/A} \quad \text{and} \quad \bar{a}_B = \cancel{\bar{a}_A} + \cancel{\bar{\alpha}_{x_1y_1z_1}\times\bar{r}_{B/A}} \times \bar{\omega}_{x_1y_1z_1}\times\left(\bar{\omega}_{x_1y_1z_1}\times\bar{r}_{B/A}\right)$$

$$\bar{a}_B = \Omega\bar{k}_1\times\left(\Omega\bar{k}_1\times\bar{r}_{B/A}\right)$$