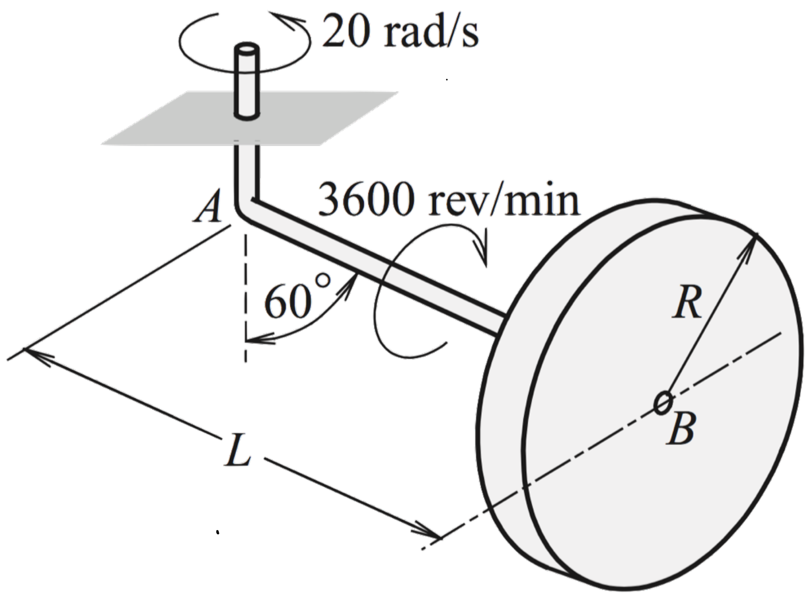
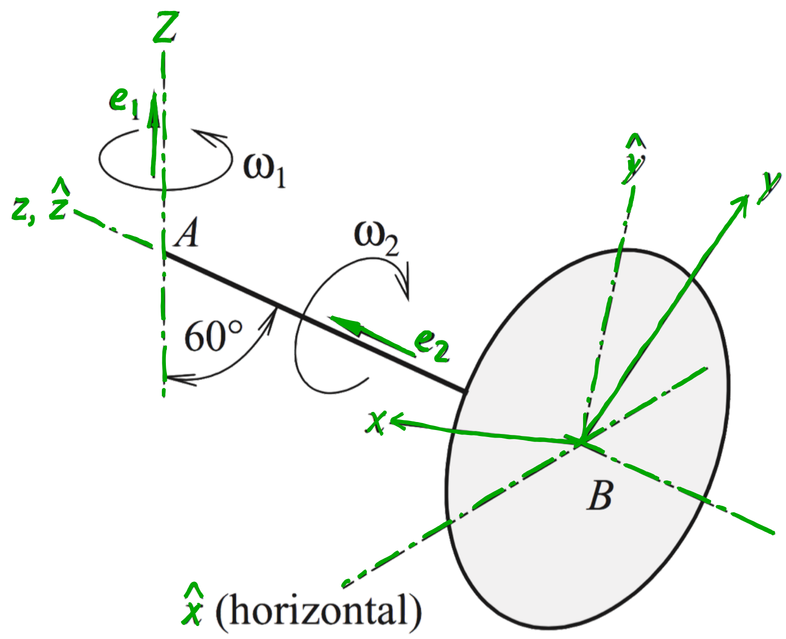
Example 3.9



Given the parameters in the figure, what are the angular velocity and angular acceleration of the disk?



XYZ - fixed axes

$\hat{x}\hat{y}\hat{z}$ - fixed to AB

xyz - rotating with disk and z aligned to AB

ω_1 rotation about $\bar{e}_1 \rightarrow \bar{e}_1 = \bar{K}$

Because $\bar{e}_1$ is fixed $\Omega_1 = 0$

ω_2 rotation about $\bar{e}_2 \rightarrow \bar{e}_2 = \bar{k} = \hat{k}$

The xyz axes undergo all the rotations that the disk does, so $\bar{\Omega}_2 = \bar{\omega}$

Total angular velocity of the disk:

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 = \omega_1 \bar{K} + \omega_2 \bar{k}$$

Total angular acceleration:

rotation rates are given as constant

$$\bar{\alpha} = \cancelto{0}{\dot{\omega}_1 \bar{e}_1} + \cancelto{0}{\omega_1 (\bar{\Omega}_1 \times \bar{e}_1)} + \cancelto{0}{\dot{\omega}_2 \bar{e}_2} + \omega_2 (\bar{\Omega}_2 \times \bar{e}_2)$$

$0 \leftarrow \Omega_1 = 0$

$$= \omega_2 (\bar{\Omega}_2 \times \bar{e}_2) = \omega_2 (\bar{\Omega}_2 \times \bar{k}) = \omega_2 (\bar{\omega} \times \bar{k})$$

Example 3.9 (cont.)

$$\bar{\omega} = \omega_1 \bar{K} + \omega_2 \bar{k} \qquad\qquad \bar{\alpha} = \omega_1\left(\bar{\omega} \times \bar{k}\right)$$

Need to choose the global coordinate system (frame to resolve all unit vectors into)

Choosing $\hat{x}\hat{y}\hat{z}$ makes it easiest to write both $\bar{e}_1$ and $\bar{e}_2$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & \sin 60^\circ \\ 0 & -\sin 60^\circ & \cos 60^\circ \end{bmatrix} \Bigg\} \text{ Rotation from } \begin{matrix} I \\ J \\ K \end{matrix} \rightarrow \begin{matrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{matrix}$$

So, $\begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} = R^T \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$ $\qquad R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60^\circ \\ 0 & \sin 60^\circ & \cos 60^\circ \end{bmatrix}$

$$\bar{K} = \left(\sin 60^\circ\right)\bar{j} + \left(\cos 60^\circ\right)\bar{k}$$

and

$$\bar{k} = \hat{k}$$

So,

$$\boxed{\bar{\omega} = \omega_1\left[\sin 60^\circ \,\bar{j} + \cos 60^\circ \,\bar{k}\right] + \omega_2 \hat{k} = \omega_1 \sin 60^\circ \,\bar{j} + \left(\omega_1 \cos 60^\circ + \omega_2\right)\bar{k}}$$

$$\boxed{\bar{\alpha} = \omega_2\left(\left[\omega_1 \sin 60^\circ \,\bar{j} + \left(\omega_1 \cos 60^\circ + \omega_2\right)\bar{k}\right] \times \bar{k}\right) = \omega_2\left(\omega_1 \sin 60^\circ \,\bar{i}\right)}$$

Velocity and Acceleration in Moving Frames (Sec. 3.5)

We found previously that

$$\bar{v}_P = \underbrace{\bar{v}_{O'}}_{\substack{\text{velocity of}\\ \text{the frame's}\\ \text{origin}}} + \underbrace{(v_P)_{xyz}}_{\substack{\text{relative velocity -}\\ \text{velocity of } P\\ \text{within the}\\ \text{frame}}} + \underbrace{\bar{\omega} \times \bar{r}_{P/O'}}_{\substack{\text{velocity of point}\\ \text{due to rotation of}\\ \text{the frame, } \bar{\omega}}}$$

$$\bar{a}_P = \frac{d\bar{v}_P}{dt} = \frac{d}{dt}\left(\bar{v}_{O'}\right) + \frac{d}{dt}\left((v_P)_{xyz}\right) + \frac{d}{dt}\left(\bar{\omega} \times \bar{r}_{P/O'}\right)$$

$$\frac{d}{dt}\left(\bar{v}_{O'}\right) = \bar{a}_{O'}$$

$$\frac{d}{dt}\left((v_P)_{xyz}\right) = \underbrace{\frac{\partial}{\partial t}(v_P)_{xyz}}_{\substack{\text{derivative assuming}\\ \text{frame is fixed}}} + \underbrace{\bar{\omega} \times (v_P)_{xyz}}_{\substack{\text{contribution from}\\ \text{rotation of frame}}} = (a_P)_{xyz} + \bar{\omega} \times (v_P)_{xyz}$$

$$\frac{d}{dt}\left(\bar{\omega} \times \bar{r}_{P/O'}\right) = \frac{d}{dt}\bar{\omega} \times \bar{r}_{P/O'} + \bar{\omega} \times \frac{d}{dt}\left(\bar{r}_{P/O'}\right)$$

$$= \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times \left((v_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}\right)$$

So,

$$\bar{a}_P = \bar{a}_{O'} + (a_P)_{xyz} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times \left(\bar{\omega} \times \bar{r}_{P/O'}\right) + 2\bar{\omega} \times \left(\bar{v}_P\right)_{xyz}$$

Velocity and Acceleration in Moving Frames (cont.)

Let's look at 2 special cases

① only translation, $\bar{\omega} = \bar{\alpha} = 0$

$$\bar{v}_P = \bar{v}_{O'} + (v_P)_{xyz} \quad \text{and} \quad \bar{a}_P = \bar{a}_{O'} + (a_P)_{xyz}$$

If $\bar{v}_{O'} = \text{constant}$, then $\bar{a}_P = (\bar{a}_P)_{xyz}$ ← We can observe total acceleration in frame xyz.

We can use this frame to formulate Newton's Laws!!!

This is an inertial or Galilean Frame

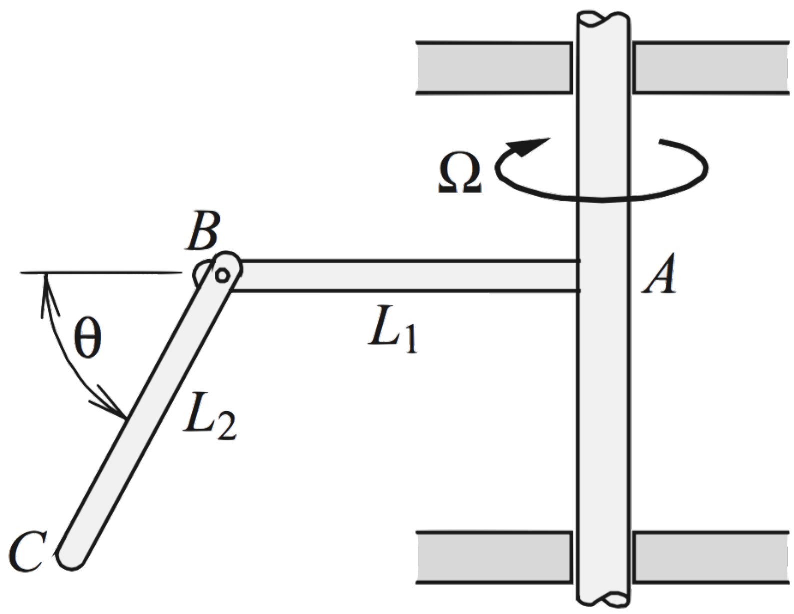
② Point P is fixed in the moving frame, $(\bar{v}_P)_{xyz} = (\bar{a}_P)_{xyz} = 0$

The frame is a rigid body and P is a point in the body

$$\bar{v}_P = \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{P/O'}$$

$$\bar{a}_P = \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'})$$

Example 3.11

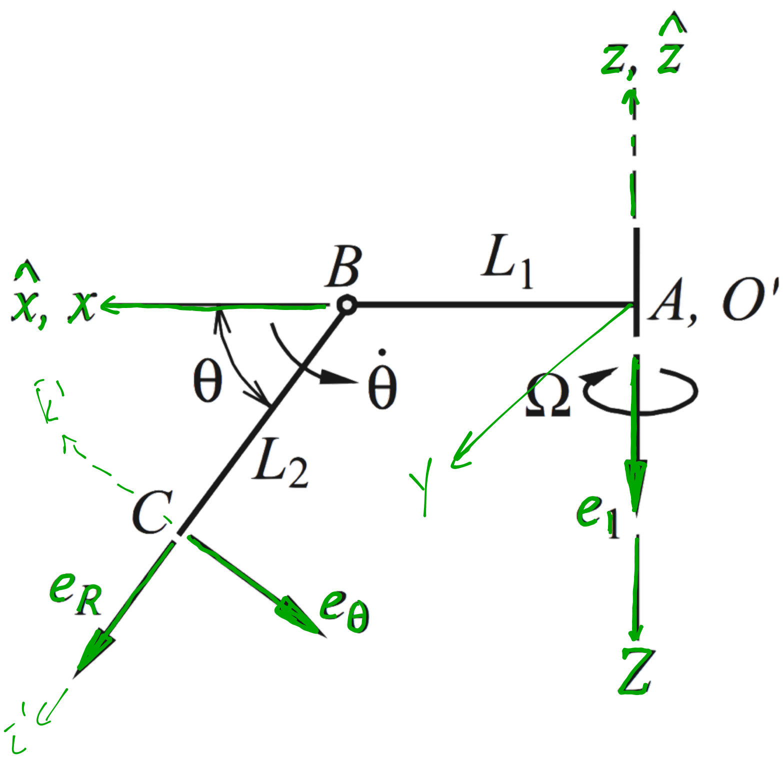


Ω = const. , but $\theta \neq$ constant

Q: What are the velocity and acceleration of C?

Book works 2 ways, using 2 different frames.

Well work only 1., slightly different than either book method. Check the book and compare.



XYZ - fixed in space

xyz - attached to T-bar, x aligned with AB

$$\bar{v}_C = \bar{v}_B + (v_C)_{xyz} + \bar{\omega}_{xyz} \times \bar{r}_{C/B}$$

$$\bar{v}_B = \cancel{\bar{v}_{O'}}^{\,0} + \cancel{(v_B)_{xyz}}^{\,0} + \bar{\omega}_{xyz} \times \bar{r}_{B/A}$$

$\bar{\omega}_{xyz}$ = angular velocity of the xyz frame

$= -\Omega \bar{k}$

$\bar{r}_{C/B} = L_2 \bar{e}_r$ ← we need to write in xyz

$\bar{e}_r = \cos\theta\,\bar{i} - \sin\theta\,\bar{k}$ ← simple rotation of θ about y axis

$\bar{e}_\theta = -\sin\theta\,\bar{i} - \cos\theta\,\bar{k}$ ← a little trickier due to $\bar{e}_\theta$ being "opposite" of $\bar{k}'$ if pure rotation

$$\bar{r}_{C/B} = L_2\cos\theta\,\bar{i} - L_2\sin\theta\,\bar{k}$$

$$(v_C)_{xyz} = \cancel{\dot{L}_2\cos\theta}^{\,0}\,\bar{i} - L_2\dot{\theta}\sin\theta\,\bar{i} - \cancel{\dot{L}_2\sin\theta}^{\,0}\,k - L_2\dot{\theta}\cos\theta\,\bar{k} = -L_2\dot{\theta}\sin\theta\,\bar{i} - L_2\dot{\theta}\cos\theta\,\bar{k}$$

Q: Why do we not need to account for $\dot{\bar{i}}$ and $\dot{\bar{k}}$ in $(v_c)_{xyz}$?

$(v_c)_{xyz}$ is the relative velocity, so we assume the frame is fixed and calculate the velocity in it.

Example 3.11 (cont.)

$\bar{\omega}_{xyz}$ = angular velocity of the frame xyz

$= -\Omega \bar{k}$

$$\bar{r}_{B/O'} = L_1 \bar{i}$$

$$\bar{v}_B = -\Omega \bar{k} \times L_1 \bar{i} = -L_1 \Omega \bar{j}$$

$$\bar{v}_C = \bar{v}_B + \left[-L_2\dot{\theta}\sin\theta\,\bar{i} - L_2\dot{\theta}\cos\theta\,\bar{k}\right] + \left[-\Omega\bar{k} \times L_2\cos\theta\,\bar{i} - L_2\sin\theta\,\bar{k}\right]$$

$$= -L_1\Omega\bar{j} + \left[-L_2\dot{\theta}\sin\theta\,\bar{i} - L_2\dot{\theta}\cos\theta\,\bar{k}\right] + \left[-L_2\Omega\cos\theta\,\bar{j}\right]$$

$$\boxed{\bar{v}_C = -L_2\dot{\theta}\sin\theta\,\bar{i} - \left(L_1 + L_2\cos\theta\right)\Omega\bar{j} - L_2\dot{\theta}\cos\theta\,\bar{k}}$$

$$\bar{a}_C = \bar{a}_B + (\bar{a}_C)_{xyz} + \cancelto{0}{\bar{\alpha}_{xyz}} \times \bar{r}_{C/B} + \bar{\omega}_{xyz} \times (\bar{\omega}_{xyz} \times \bar{r}_{C/B}) + 2\bar{\omega}_{xyz} \times (\bar{v}_C)_{xyz}$$

$$\bar{a}_B = \cancelto{0}{\bar{a}_{O'}} + \cancelto{0}{(\bar{a}_B)_{xyz}} + \cancelto{0}{\bar{\alpha}_{xyz}} \times \bar{r}_{B/O'} + \bar{\omega}_{xyz} \times (\bar{\omega}_{xyz} \times \bar{r}_{B/O'}) + \cancelto{0}{2\bar{\omega}_{xyz} \times (\bar{v}_B)_{xyz}}$$

$$\bar{\alpha}_{xyz} = \frac{d\bar{\omega}_{xyz}}{dt} = -\cancelto{0,\ \Omega = \text{const}}{\dot{\Omega}}\,\bar{k} + \cancelto{0,\ \bar{k}\text{ constant direction}}{\Omega\dot{\bar{k}}} \qquad \bar{\alpha}_{xyz} = 0$$

We can already write $\bar{a}_B$

$$\bar{a}_B = \bar{\omega}_{xyz} \times \underbrace{(\bar{\omega}_{xyz} \times \bar{r}_{B/O'})}_{\text{We know this from } \bar{v}_B \text{ calc} = L_1\Omega\bar{j}} = -\Omega\bar{k} \times \left(-\Omega\bar{k} \times L_1\bar{i}\right) = -\Omega\bar{k} \times \left(-L_1\Omega\bar{j}\right) = -L_1\Omega^2\bar{i}$$

$$(\bar{a}_C)_{xyz} = -L_2\dot{\theta}^2\bar{e}_r + L_2\ddot{\theta}\,\bar{e}_\theta$$ ← shortcut is to remember cylindrical/polar coord. accel.

$$= -L_2\dot{\theta}^2\left[\cos\theta\,\bar{i} - \sin\theta\,\bar{k}\right] + L_2\ddot{\theta}\left[-\sin\theta\,\bar{i} - \cos\theta\,\bar{k}\right]$$

$$= \left[-L_2\dot{\theta}^2\cos\theta - L_2\ddot{\theta}\sin\theta\right]\bar{i} + \left[L_2\dot{\theta}^2\sin\theta - L_2\ddot{\theta}\cos\theta\right]\bar{k}$$

Example 3.11 (cont.)

$$\underbrace{\bar{\omega}_{xyz} \times (\bar{\omega}_{xyz} \times \bar{r}_{C/B})} = -\Omega \bar{k} \times \left[-\Omega \bar{k} \times L_2 \cos\theta \,\bar{i} - L_2 \sin\theta \,\bar{k} \right]$$

$$= -\Omega \bar{k} \times -L_2 \Omega \cos\theta \,\bar{j} = -L_2 \Omega^2 \cos\theta \,\bar{i}$$

We again know this from $\bar{v}_C$ calc. $= -L_2 \Omega \cos\theta \,\bar{j}$

$$2\bar{\omega}_{xyz} \times (\bar{v}_C)_{xyz} = -2\Omega \bar{k} \times \left[-L_2 \dot{\theta} \sin\theta \,\bar{i} - L_2 \dot{\theta} \cos\theta \,\bar{k} \right] = 2L_2 \dot{\theta} \Omega \sin\theta \,\bar{j}$$

Now, just fill in all the terms:

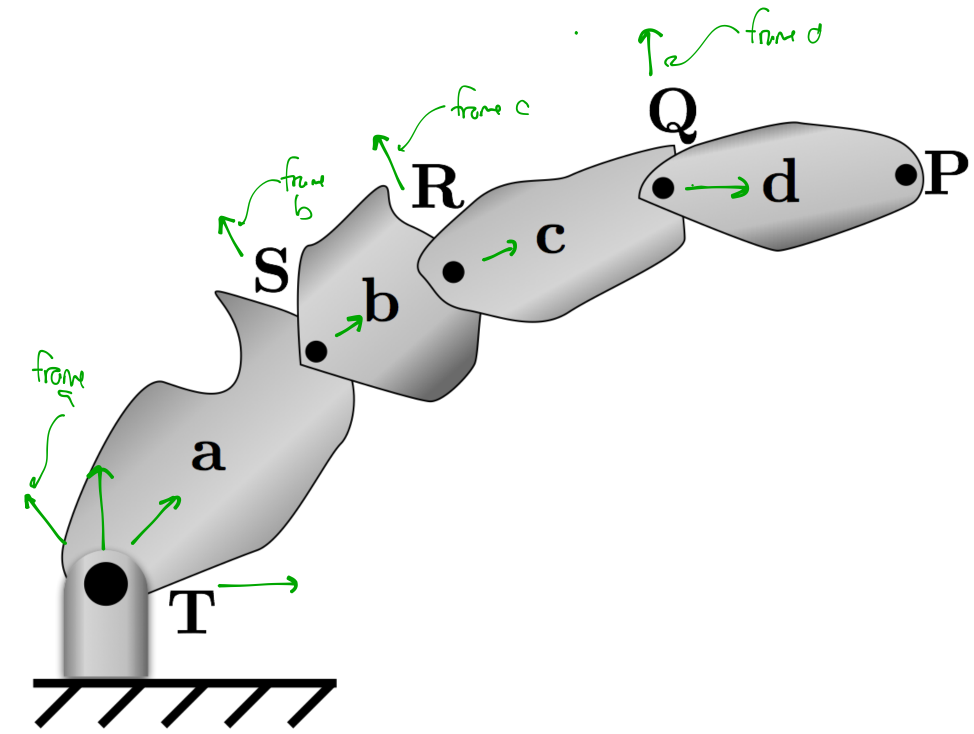
$$\bar{a}_C = [\bar{a}_B] + [(\bar{a}_C)_{xyz}] + [\bar{\omega}_{xyz} \times (\bar{\omega}_{xyz} \times \bar{r}_{C/B})] + [2\bar{\omega}_{xyz} \times (\bar{v}_C)_{xyz}]$$

$$\bar{a}_C = [L_1 \Omega^2 \bar{i}] + \left[\left[-L_2 \dot{\theta}^2 \cos\theta - L_2 \ddot{\theta} \sin\theta \right] \bar{i} + \left[L_2 \dot{\theta}^2 \sin\theta - L_2 \ddot{\theta} \cos\theta \right] \bar{k} \right]$$

$$+ \left[-L_2 \Omega^2 \cos\theta \,\bar{i} \right] + \left[2L_2 \dot{\theta} \Omega \sin\theta \,\bar{j} \right]$$

$$\boxed{\bar{a}_C = \left[-L_2 \dot{\theta}^2 \cos\theta - L_2 \ddot{\theta} \sin\theta - (L_1 + L_2 \cos\theta)\Omega^2 \right] \bar{i} + \left[2L_2 \dot{\theta} \Omega \sin\theta \right] \bar{j} + \left[L_2 \dot{\theta}^2 \sin\theta - L_2 \ddot{\theta} \cos\theta \right] \bar{k}}$$

"Linking" Frames



We can "link" a series of moving reference frames

$$\bar{v}_P = \bar{v}_Q + (v_P)_d + (\bar{\omega}_d \times \bar{r}_{P/Q})$$

$$\bar{v}_Q = \bar{v}_R + (v_Q)_c + (\bar{\omega}_c \times \bar{r}_{Q/R})$$

$$\bar{v}_R = \bar{v}_S + \dots$$

$$\bar{v}_S = v_T + \dots$$

These terms = 0 if both points are on a rigid body

Can follow a similar process for acceleration.