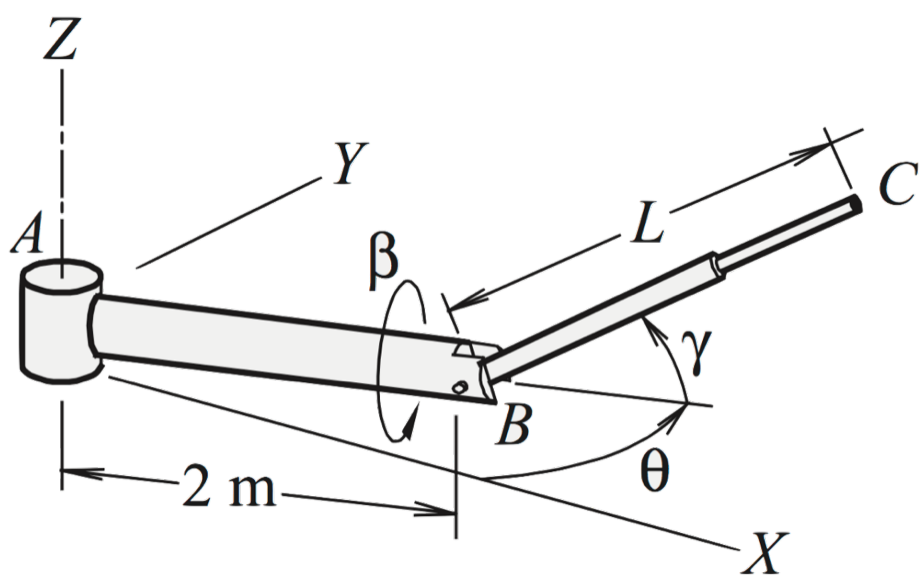
Example 3.6



AB rotates by θ about Z

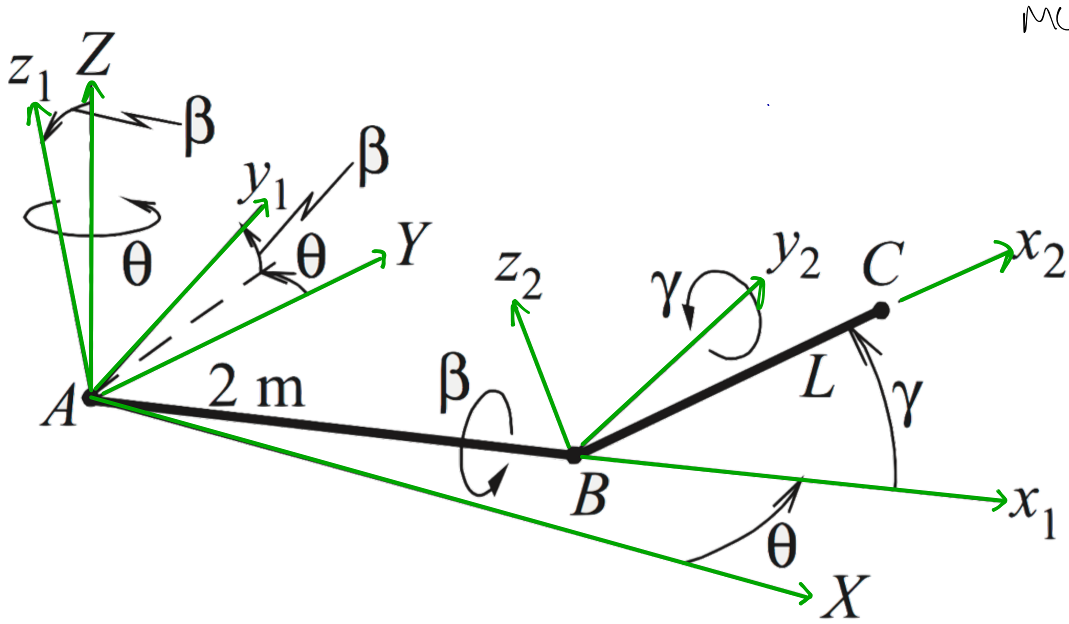
BC rotates by β about AB

BC rotates by γ about pin at B

Move: $\theta = 50^\circ$, $\beta = 30^\circ$, $\gamma = 60^\circ$

$L_0 = 0.5\text{m} \rightarrow L_f = 1.5\text{m}$

What is displacement of C $\Delta\bar{r}_C$ from these moves?



Define $x_1y_1z_1$ fixed to AB

$x_2y_2z_2$ fixed to BC

$z_1 = K$ when $\beta = 0$ and y_2 is aligned with pin at B

We'll solve by solving for displacement of B in $x_1y_1z_1$, then for C in $x_2y_2z_2$

$$\Delta\bar{r}_B = \Delta\bar{r}_A + R_{1f}\left(\Delta\bar{r}_B\right)_{xyz} + \left(R_{1f} - R_{10}\right)\begin{bmatrix} x_{1B} \\ y_{1B} \\ z_{1B} \end{bmatrix}_0$$

$$\Delta\bar{r}_C = \Delta\bar{r}_B + R_{2f}\left(\Delta\bar{r}_C\right)_{xyz} + \left(R_{2f} - R_{20}\right)\begin{bmatrix} x_{2C} \\ y_{2C} \\ z_{2C} \end{bmatrix}_0$$

Initially, there is no rotation and $x_1y_1z_1$, $x_2y_2z_2$, and XYZ are aligned, so

$R_{10} = R_{20} = I$ (U in book = still Identity matrix)

Example 3.6 (cont.)

Q: What is the rotation matrix from XYZ to $x_1y_1z_1$

① Rotation of θ about Z

② Rotation of β about x_1

$$R_{1f} = R_\beta R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q: Rotation of $x_2y_2z_2$ relative to XYZ?

Just a rotation of $-\gamma$ about y_2 relative to $x_1y_1z_1$, so

$$R_{2f} = R_\gamma R_\beta R_\theta = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, all that's left are the relative position + displacements

$\Delta\bar{r}_A = 0$

$\left(\bar{r}_{B/A}\right)_0 = 2\bar{i}_1$ ← B is 2m along $\bar{i}$ from A

$\left(\Delta\bar{r}_B\right)_{x_1y_1z_1} = 0$ ← It doesn't change position in the $x_1y_1z_1$ frame

$\left(\bar{r}_{C/B}\right)_0 = 0.5m\,\bar{i}_2$ ← C starts 0.5m from B along $\bar{i}_2$

$\left(\Delta\bar{r}_C\right)_{x_2y_2z_2} = 1m\,\bar{i}_2$ ← C moves from 0.5m to 1.5m as rotating

Time Derivatives (Sec. 3.3)

Now, look at an infinitesimally small displacement ⇒ infinitesimal small amount of time

So

$$\Delta \bar{r}_p \rightarrow d\bar{r}_p \qquad \Delta \bar{r}_{0'} \rightarrow d\bar{r}_{0'} \qquad (\Delta \bar{r}_p)_{xyz} \rightarrow (d\bar{r}_p)_{xyz}$$

Q: How can we write the rotation matrices?

$R_f = R_z R_y R_x R_0$ $\qquad$ $R_z R_y R_x$ - sequence of body-fixed rotations from R_0 to R_f

If these angles are infinitesimal, the $\theta_i \rightarrow d\theta_i$

For the rotation matrix $\cos d\theta_i = 1$ and $\sin d\theta_i = d\theta_i$ so

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d\theta_x \\ 0 & -d\theta_x & 1 \end{bmatrix} \qquad R_y = \begin{bmatrix} 1 & 0 & -d\theta_y \\ 0 & 1 & 0 \\ d\theta_y & 0 & 1 \end{bmatrix} \qquad R_z = \begin{bmatrix} 1 & d\theta_z & 0 \\ -d\theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_z R_y R_x = \begin{bmatrix} 1 & d\theta_z & -d\theta_y \\ -d\theta_z & 1 & d\theta_z \\ d\theta_y & -d\theta_x & 1 \end{bmatrix} \quad d\theta_i d\theta_j \approx 0$$

We can find:

$$\begin{bmatrix} d\bar{r}_p \cdot \bar{i} \\ d\bar{r}_p \cdot \bar{j} \\ d\bar{r}_p \cdot \bar{k} \end{bmatrix} = \begin{bmatrix} d\bar{r}_{0'} \cdot \bar{i} \\ d\bar{r}_{0'} \cdot \bar{j} \\ d\bar{r}_{0'} \cdot \bar{k} \end{bmatrix} + \begin{bmatrix} (d\bar{r}_p)_{xyz} \cdot \bar{i} \\ (d\bar{r}_p)_{xyz} \cdot \bar{j} \\ (d\bar{r}_p)_{xyz} \cdot \bar{k} \end{bmatrix} + \begin{bmatrix} 0 & -d\theta_z & d\theta_y \\ d\theta_z & 0 & -d\theta_x \\ -d\theta_y & d\theta_x & 0 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

no f b/c infinitesimal motion means $(\bar{r}_p)_0 = (\bar{r}_p)_f$

Matrix form of:

$$d\bar{r}_p = d\bar{r}_{0'} + (d\bar{r}_p)_{xyz} + d\bar{\theta} \times \bar{r}_{p/0'} \qquad \text{where } \underbrace{d\bar{\theta} = \theta_x \bar{i} + \theta_y \bar{j} + \theta_z \bar{k}}_{\text{infinitesimal rotation vector}}$$

Time Derivatives (cont.)

$$d\bar{r}_P = d\bar{r}_{O'} + (d\bar{r}_P)_{xyz} + d\bar{\Theta} \times \bar{r}_{P/O'}$$

We know that $\bar{v} = \frac{d\bar{r}}{dt}$, so dividing this equation by dt gives us $\bar{v}_P$

$$\bar{v}_P = \bar{v}_{O'} + (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}$$

$\bar{\omega} \equiv$ angular velocity of frame xyz

$(v_P)_{xyz} \equiv$ relative velocity - velocity in xyz frame

Note that this is equivalent to

$$\bar{v}_P = \bar{v}_{O'} + \frac{d}{dt}\left(\bar{r}_{P/O'}\right) \rightarrow \underbrace{\frac{d}{dt}\left(\bar{r}_{P/O'}\right) = (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}}$$

This is generally true for vectors in moving frames

$$\dot{\bar{A}} = \underbrace{\frac{\partial \bar{A}}{\partial t}}_{\text{derivative if the frame is fixed}} + \underbrace{\bar{\omega}}_{\text{angular velocity of the frame}} \times \bar{A}$$

Q: What does this mean for the derivatives of unit vectors

$$\bar{e} \rightarrow \dot{\bar{e}} = \underbrace{\cancel{\frac{\partial \bar{e}}{\partial t}}}_{0} + \bar{\omega} \times \bar{e}$$

The length of $\bar{e}$ is constant

Angular Velocity and Acceleration (Sec. 3.4)

If angular velocity is $\bar{\omega}$, then angular acceleration is $\bar{\alpha} = \underbrace{\frac{d\bar{\omega}}{dt}}_{\text{Total time deriv.}}$

$\bar{\omega} = \sum \omega_n \bar{e}_n$ ← sum of simple rotations about axes $\bar{e}_n$ ← Define $e_n = \bar{i}_n, \bar{j}_n,$ or $\bar{k}_n$

Define $\bar{\Omega}_n$ as the angular velocity of frame n $(x_n y_n z_n)$

$$\bar{\alpha} = \sum \left(\dot{\omega}_n \bar{e}_n + \omega_n \dot{\bar{e}}_n \right)$$

Q: What is $\dot{\bar{e}}_n$?

$$\dot{\bar{e}}_n = \Omega_n \times \bar{e}_n$$

So,

$$\bar{\alpha} = \sum \Big(\underbrace{\dot{\omega}_n \bar{e}_n}_{\text{change in rotation rate}} + \underbrace{\omega_n (\bar{\Omega}_n \times \bar{e}_n)}_{\text{change in orientation of } \bar{e}_n} \Big)$$

can have angular acceleration even if $\dot{\omega}_n = 0$ (constant rotation rate)

- This term almost never shows up in planar cases seen in undergrad dynamics
- Intuition from planar case can be wrong because it ignores this term

Procedure to Find Angular Velocity and Acceleration (Sec 3.4.2)

- Examine rotation, write it as a series of simple rotations, ω_n
- For each ω_n, define a frame where ω_n aligns with $\bar{i}_n, \bar{j}_n$, or $\bar{k}_n$
- Sum rotations into total angular velocity vector

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \ldots + \omega_n \bar{e}_n$$

- Find Ω_n, from angular velocity, using a similar method
- Solve for $\bar{\alpha}$ by taking the full time deriv.

$$\bar{\alpha} = \underbrace{\dot{\omega}_1 \bar{e}_1}_{\substack{\text{Accel from} \\ \text{change of} \\ \omega_1}} + \underbrace{\omega_1 (\bar{\Omega}_1 \times \bar{e}_1)}_{\substack{\text{Accel from} \\ \text{change of} \\ \bar{e}_1 \text{ direction}}} + \underbrace{\dot{\omega}_2 \bar{e}_2}_{\substack{\text{Accel from} \\ \text{change of} \\ \omega_2}} + \underbrace{\omega_2 (\Omega_2 \times \bar{e}_2)}_{\substack{\text{Accel from} \\ \text{change of} \\ \bar{e}_2 \text{ direction}}} + \ldots + \underbrace{\dot{\omega}_n \bar{e}_n}_{\substack{\text{Accel from} \\ \text{change of} \\ \omega_2}} + \underbrace{\omega_n (\Omega_n \times \bar{e}_n)}_{\substack{\text{Accel from change} \\ \text{of } \bar{e}_n \text{ direction}}}$$

Keys: · must write all vectors in same frame before summing

· choice of frames can simplify the "math" of the procedure