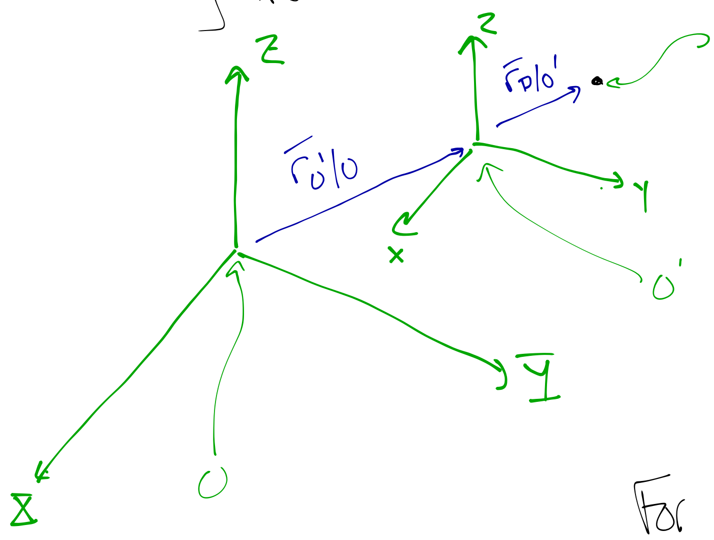
Chapter 3 - Relative Motion

Coordinate Transforms (Sec. 3.1)

Generally, measured w.r.t.

- fixed frame → absolute
- moving frame → relative



$$\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}$$

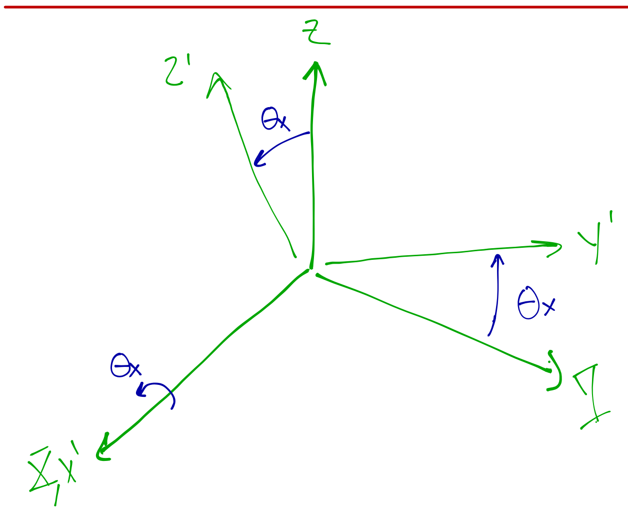
To be useful, we need to describe these vectors with a common set of unit vectors.

For example, if $\bar{I}\bar{J}\bar{K}$ is aligned with $\bar{i}\bar{j}\bar{k}$, then motion is only translational and

$$\bar{I} = \bar{i}, \quad \bar{J} = \bar{j}, \quad \text{and} \quad \bar{K} = \bar{k}$$

So
$$\bar{r}_{P/O} = \left(X_{O'}\bar{I} + Y_{O'}\bar{J} + Z_{O'}\bar{K}\right) + \left(x_P\bar{i} + y_P\bar{j} + z_P\bar{k}\right)$$
$$= \left(X_{O'} + x_P\right)\bar{I} + \left(Y_{O'} + y_P\right)\bar{J} + \left(Z_{O'} + z_P\right)\bar{K}$$

Rotational Transformations (Sec. 3.1.1)

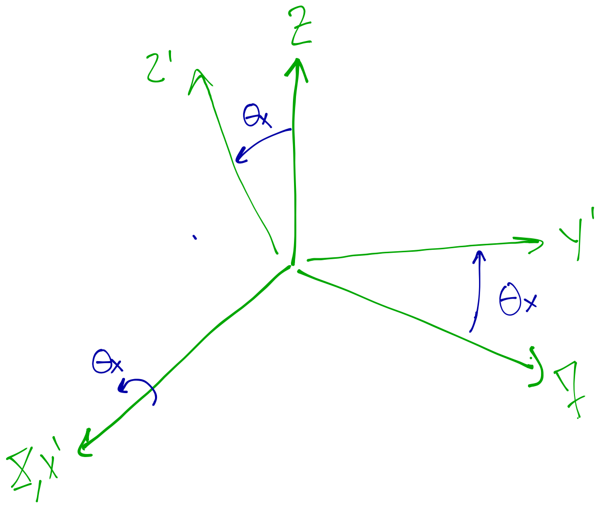


Want to write a Rotation Transformation Matrix, R, to relate XYZ and x'y'z', such that

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} = R \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} \quad \underline{\text{and}} \quad \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} = R^T \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix}$$

See book for general derivation. We'll focus on the common cases.

Rotation Sequences (Sec. 3.1.2)



Let's look at this case

$\bar{i}' = \bar{i}$

$\bar{j}' = \cos\theta_x \bar{j} + \sin\theta_x \bar{k}$

$\bar{k}' = -\sin\theta_x \bar{j} + \cos\theta_x \bar{k}$

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}}_{R_x} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix}$$

$R_x \leftarrow$ rotation matrix for simple rotations about $\bar{i}$-axis

$$R_y = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

$R_y \leftarrow$ rotation matrix for simple rotations about $\bar{j}$-axis

$$R_z = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_z \leftarrow$ rotation matrix for simple rotations about $\bar{k}$-axis

Due to trig properties, rotating by a negative angle = transpose of positive equiv

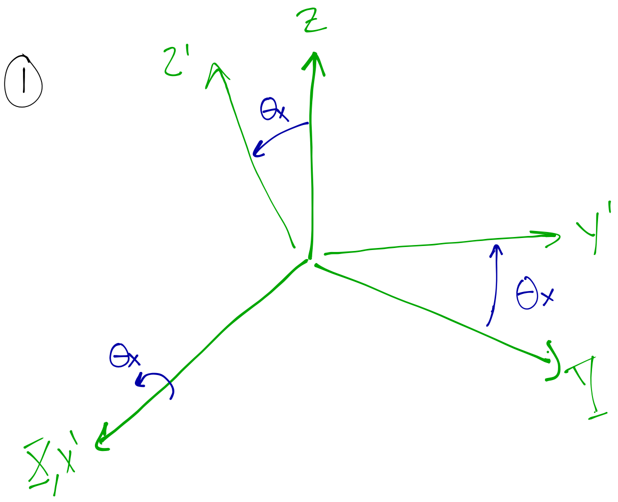
$$R_x(-\theta_x) = [R_x(\theta_x)]^T, \quad R_y(-\theta_y) = [R_y(\theta_y)]^T, \quad \text{and} \quad R_z(-\theta_z) = [R(\theta_z)]^T$$

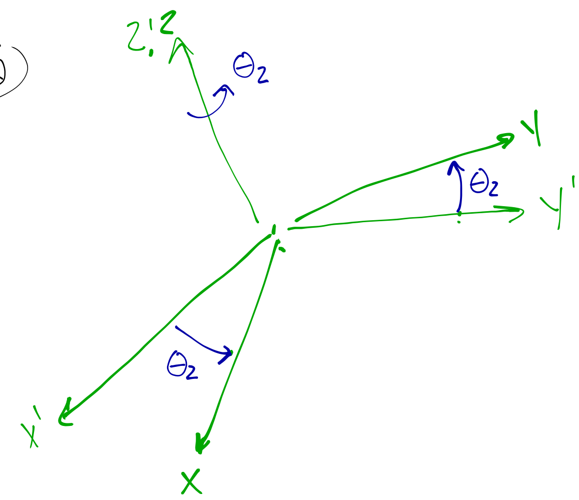
Body-fixed Rotations

Body-fixed coordinate systems → fixed within a body

Body-fixed rotations → rotation about a body-fixed coordinate

Let's look at a sequence of body-fixed rotations

① 

② 

① Rotation of θ_x about X, x' axis

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = R_x \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}$$

② Rotation of θ_z about z', z axis

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = R_z \begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i' \\ j' \\ k' \end{bmatrix}$$

Body-fixed Rotations (cont.)

(3) (1), then (2)

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = R_2 \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ k' \end{bmatrix} = R_2 R_x \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} \qquad R_{TOT} = R_2 R_x$$

For body-fixed rotations, this pattern holds. Pre-multiply by each rotation

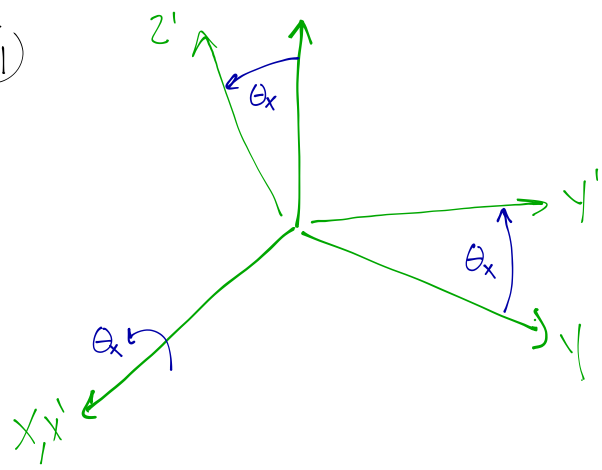
So for N body-fixed rotations

$$R = [R_N] \ldots [R_2][R_1]$$

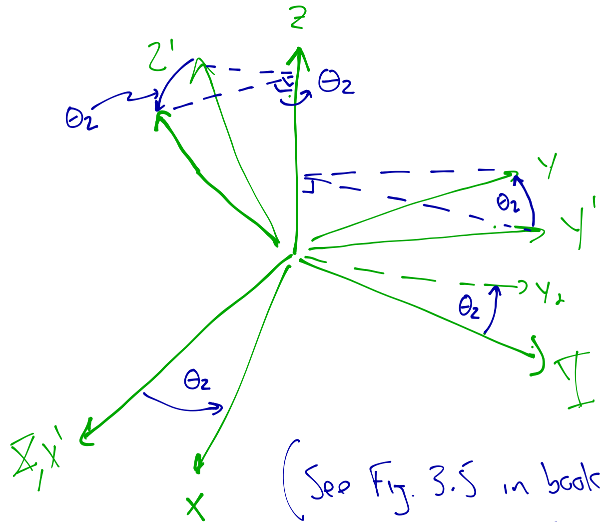
Space-fixed Rotations

Rotations are about a fixed axis in space, rather than one fixed to the body

(1)



(2)



rotate x'y'z' to xyz

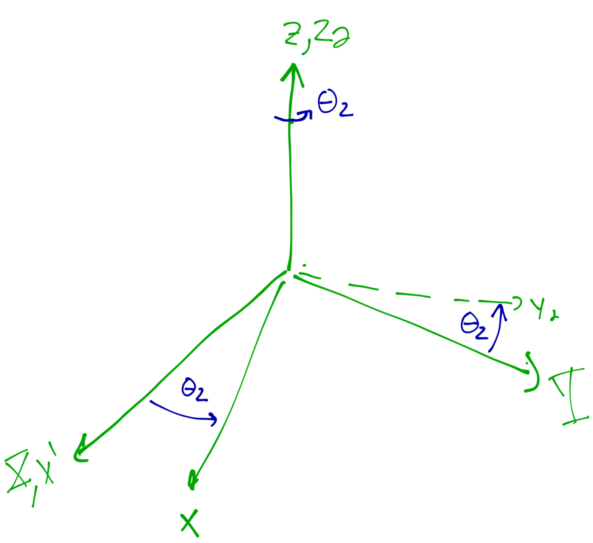
(See Fig. 3.5 in book for a better version of this figure.)

(1) Still the same. In this case, the body-fixed and space-fixed axes are aligned

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = R_x \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}$$

(2) About fixed ("original") Z axis.

Let's look at $x_2y_2z_2$ – initially aligned with XYZ, rotates about Z, z_2 axis



$$\begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} = R_z \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}$$

Key point: relative position of x'y'z' and $x_2y_2z_2$ don't change during the space fixed rotation. So, since R_x gave us the transformation from XYZ to x'y'z', it also gives us $x_2y_2z_2 \to xyz$

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = R_x \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix}$$

Space-fixed Rotations (cont.)

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = R_x \begin{bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{bmatrix} = R_z \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = \underbrace{R_x R_z}_{R} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}$$

R ← post mult. by space-fixed rotation

↑

Holds true for all space-fixed rotations

For N space-fixed Rotations, form the total rotation matrix by post-mult. by each in the sequence

$$R = [R_1][R_2]\ldots[R_N]$$

For sequences of body-fixed and space-fixed rotations, just pre-mult by body-fixed and post-mult by space-fixed in the order in which they are applied.

Total rotation matrix depends on the order in which rotations occur.

Q: What about a rotation about an arbitrary axis?

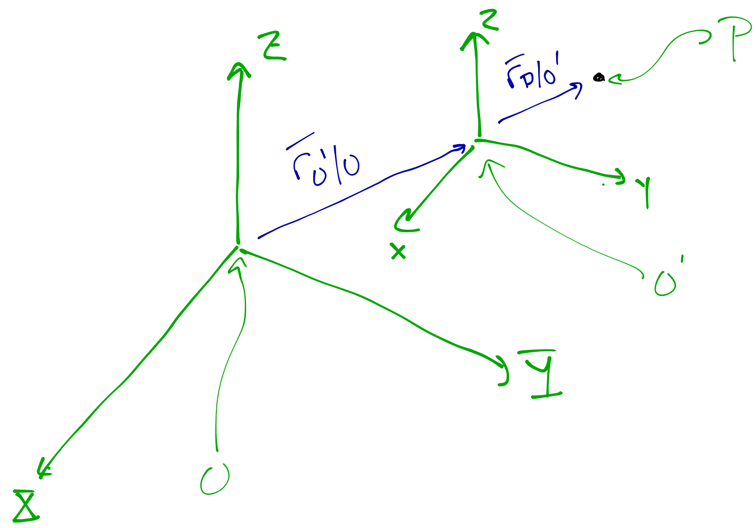
high-level idea - ① rotate such that a body fixed axis aligns with rotation

② Complete the rotation

③ "undo" the rotation from ① Q: How? → R^T

Displacement (Sec. 3.2)

We saw before:

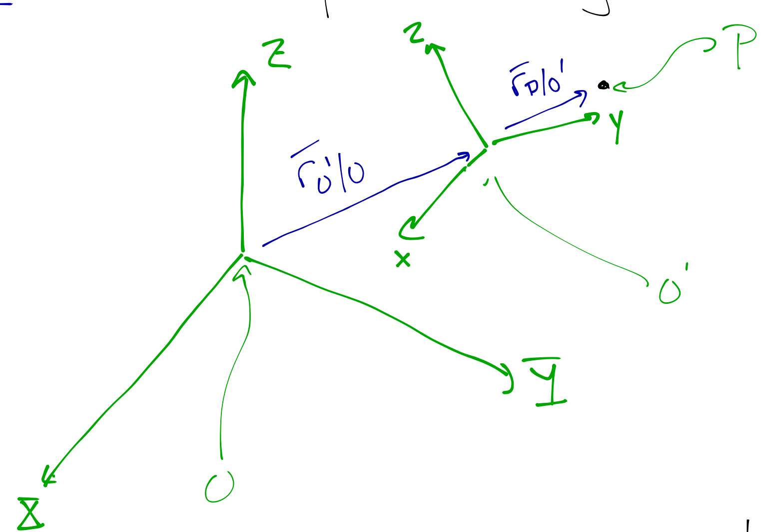


$$\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}$$

Because xyz was aligned to XYZ,

$$\bar{r}_{P/O} = \left(X_{O'}\bar{I} + Y_{O'}\bar{J} + Z_{O'}\bar{K} \right) + \left(x_p \bar{i} + y_p \bar{j} + z_p \bar{k} \right)$$
$$= \left(X_{O'} + x_p \right)\bar{I} + \left(Y_{O'} + y_p \right)\bar{J} + \left(Z_{O'} + z_p \right)\bar{K}$$

Q: What if xyz is not aligned to XYZ?



We need to rotate ijk to align with IJK

$$\bar{r}_{O'/O} = X_{O'}\bar{I} + Y_{O'}\bar{J} + Z_{O'}\bar{K}$$
$$\bar{r}_{P/O'} = x_p \bar{i} + y_p \bar{j} + z_p \bar{k}$$

If R is defined such that

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = R \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}$$

then

$$\bar{r}_{P/O} = \left[X_{O'}\bar{I} + Y_{O'}\bar{J} + Z_{O'}\bar{K} \right] + \left[x_p \bar{i} + y_p \bar{j} + z_p \bar{k} \right]$$

We can write in matrix/vector form:

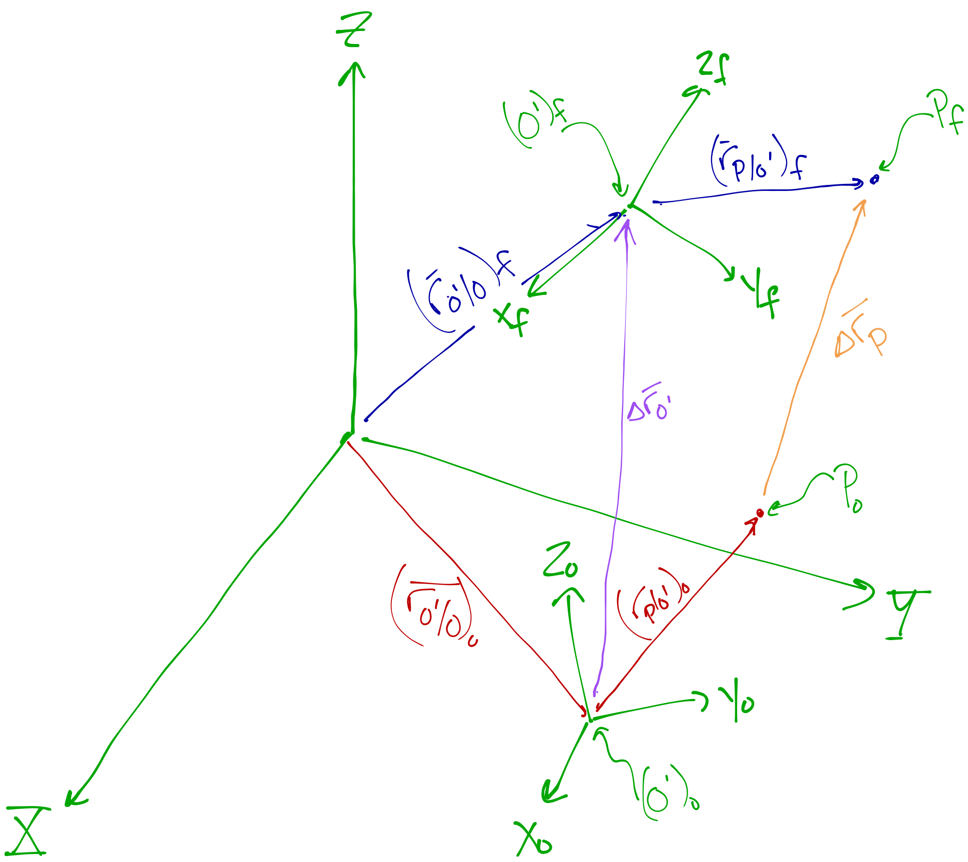
$$\bar{r}_{P/O} = \begin{bmatrix} X_{O'} \\ Y_{O'} \\ Z_{O'} \end{bmatrix}_{XYZ} + \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_{xyz}$$

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_{xyz} = \left[R \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} \right]_{XYZ}$$

$$\bar{r}_{P/O} = \begin{bmatrix} X_{O'} \\ Y_{O'} \\ Z_{O'} \end{bmatrix} + R \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

Displacement (cont.)

Now let's look at the displacement of a point



$$\Delta\bar{r}_P = \underbrace{(\bar{r}_{P/O})_f}_{\text{final pos.}} - \underbrace{(\bar{r}_{P/O})_o}_{\text{original pos.}}$$

$$(\bar{r}_{P/O})_o = (\bar{r}_{O'/O})_o + (\bar{r}_{P/O'})_o = \begin{bmatrix} X_{O'} \\ Y_{O'} \\ Z_{O'} \end{bmatrix}_o + R_o \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}_o$$

$$(\bar{r}_{P/O'})_f = (\bar{r}_{O'/O})_f + (\bar{r}_{P/O'})_f = \begin{bmatrix} X_{O'} \\ Y_{O'} \\ Z_{O'} \end{bmatrix}_f + R_f \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}_f$$

Both $(O')_o$ and $(O')_f$ are defined in the fixed inertial XYZ frame, so these can add direction – these terms are $\Delta\bar{r}_{O'}$

$$\Delta\bar{r}_P = \Delta\bar{r}_{O'} + R_f \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}_f - R_o \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}_o$$

Define the relative displacement to be $(\Delta\bar{r}_P)_{xyz}$ ← What the displacement would be to someone fixed in xyz fra

Will see $\Delta\bar{r}_P$ called absolute displacement

[It is $(\Delta\bar{r}_P)_{XYZ}$]

Displacement (cont.)

Let's look at the displacement from this perspective.

$$\left(\Delta \bar{r}_P\right)_{xyz} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_f - \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_o$$

These are in the same frame, xyz, which is moving relative to XYZ

Rearranging terms.

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_f = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_o + \begin{bmatrix} (\Delta \bar{r}_P)_{xyz} \cdot \bar{i} \\ (\Delta \bar{r}_P)_{xyz} \cdot \bar{j} \\ (\Delta \bar{r}_P)_{xyz} \cdot \bar{k} \end{bmatrix}$$

Substitute this into

$$\Delta \bar{r}_P = \Delta \bar{r}_{O'} + R_f \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_f - R_o \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_o \longrightarrow \Delta \bar{r}_P = \Delta \bar{r}_{O'} + R_f \left[\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_o + \begin{bmatrix} (\Delta \bar{r}_P)_{xyz} \cdot \bar{i} \\ (\Delta \bar{r}_P)_{xyz} \cdot \bar{j} \\ (\Delta \bar{r}_P)_{xyz} \cdot \bar{k} \end{bmatrix} \right] - R_o \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_o$$

Collect terms:

$$\Delta \bar{r}_P = \underbrace{\Delta \bar{r}_{O'}}_{\text{Translational Displacement}} + \underbrace{R_f \begin{bmatrix} (\Delta \bar{r}_P)_{xyz} \cdot \bar{i} \\ (\Delta \bar{r}_P)_{xyz} \cdot \bar{j} \\ (\Delta \bar{r}_P)_{xyz} \cdot \bar{k} \end{bmatrix}}_{\text{relative displacement of p within moving frame}} + \underbrace{\left[R_f - R_o\right] \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}_o}_{\text{Rotational displacement of the frame}}$$

3 components contribute to displacements in terms of moving frames