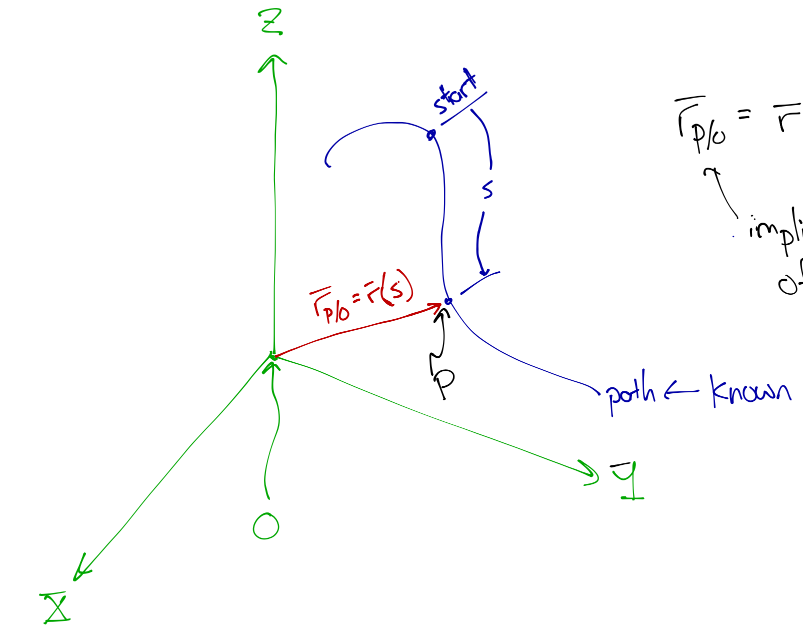
Chapter 2 - Particle Kinematics

Path Variables (Sec 2.1) ← Also called intrinsic coordinates



$\bar{r}_{P/O} = \bar{r}(s)$ and $s(t)$

implicit function of time, because s is a function of time

$+s$ = "forward" along path
$-s$ = "backward" along path

Q: What's the velocity of point P?

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \dot{s}\underbrace{\frac{d\bar{r}}{ds}}$$

determined by the position dependence on path

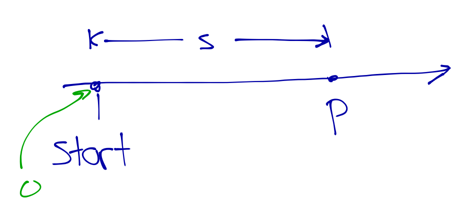
Q: What about acceleration of P?

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}\left(\dot{s}\frac{d\bar{r}}{ds}\right) = \ddot{s}\frac{d\bar{r}}{ds} + \dot{s}\frac{d}{dt}\underbrace{\left(\frac{d\bar{r}}{ds}\right)} = \ddot{s}\frac{d\bar{r}}{ds} + \dot{s}^2\underbrace{\left(\frac{d^2\bar{r}}{ds^2}\right)}$$

determined solely by s

Another property of the path

Look at a straight line in direction $\bar{e}$, so



$\bar{r}_{P/O} = s\bar{e}$,

$\frac{d\bar{r}}{ds} = \bar{e}$

$\bar{e}$'s direction is constant, so $\frac{d^2\bar{r}}{ds^2} = 0$

$\bar{v} = \dot{s}\bar{e}$ and $\bar{a} = \ddot{s}\bar{e}$

only aligned here b/c $\bar{e}$ = const in this problem

Not generally true!

Tangent and Normal Components (Sec. 2.1.1)

Define $\bar{e}_t$ as the direction tangent to the path at any point along it

$$\bar{e}_t = \frac{d\bar{r}}{ds} \quad \leftarrow \text{defines unit vector tangent to path}$$

Now, look at the $\frac{d^2\bar{r}}{ds^2}$ from the earlier equation:

$$\frac{d^2\bar{r}}{ds^2} = \frac{d\bar{e}_t}{ds}$$

Q: What's the relationship of directions $\bar{e}_t$ and $\frac{d\bar{e}_t}{ds}$?

$\perp \rightarrow \bar{e}_t \cdot \frac{d\bar{e}_t}{ds} = 0$

So, use this to define the normal direction

$$\underbrace{\bar{e}_n = \rho \frac{d\bar{e}_t}{ds}}_{\text{points toward "center" of curve}} \qquad \rho \equiv \frac{1}{\left|\frac{d\bar{e}_t}{ds}\right|} \leftarrow \text{needed to scale } \bar{e}_n \text{ to a unit vector}$$

Q: What is this physically?

radius of curvature

Let's use these unit vectors to rewrite the particle velocity and accel:

$\bar{v} = v\bar{e}_t$, where $v = \dot{s}$ ← speed

scalar

$\bar{a} = \dot{v}\bar{e}_t + \frac{v^2}{\rho}\bar{e}_n$

These two vectors always lie in the osculating plane formed by $\bar{e}_t$ and $\bar{e}_n$

Note: The osculating plane is only constant if the path is planar

Define a 3rd unit vector in the binormal direction:

$$\bar{e}_b = \bar{e}_t \times \bar{e}_n$$

Tangent and Normal Components (cont.)

We can use Newton's 2nd Law with these coords

Key point: We can pick the coordinate system used to describe systems. A smart choice can save a lot of work.

$$\sum \bar{F} = m\bar{a}$$ all components must be equal so

$$\sum \bar{F} = m\left[\dot{v}\bar{e}_t + \frac{v^2}{\rho}\bar{e}_n\right]$$

$$\sum \bar{F}_t = \sum \bar{F}\cdot\bar{e}_t = m\dot{v}$$ tangential

$$\sum F_n = \sum \bar{F}\cdot\bar{e}_n = m\frac{v^2}{\rho}$$ normal

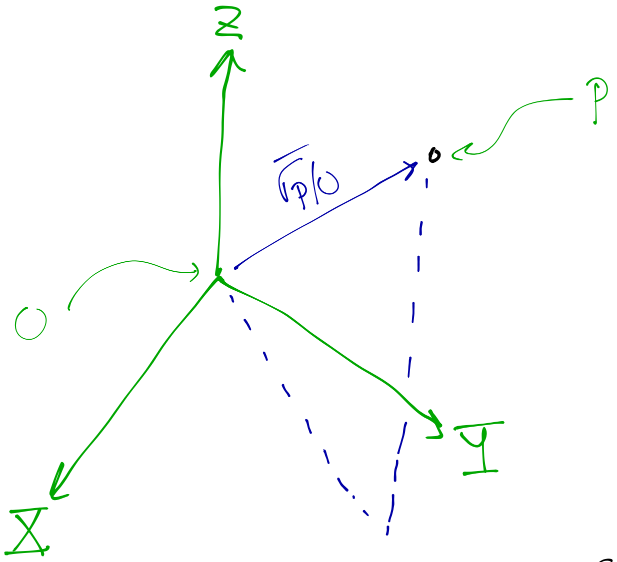
$$\sum F_b = \sum \bar{F}\cdot\bar{e}_b = 0$$ binormal

usually forces in these directions will be constraint forces, keeping the particle on the path

Review Sections 2.1.2 and 2.1.3 on your own for now.
We may return to them later.

Rectangular Cartesian Coordinates (Sec 2.2)

extrinsic coords – properties are independent of path



$$\bar{r}_{P/O} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$

unit vectors $\bar{i}, \bar{j}, \bar{k}$ are all constant so

$$\bar{v}_P = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} = v_x\bar{i} + v_y\bar{j} + v_z\bar{k}$$

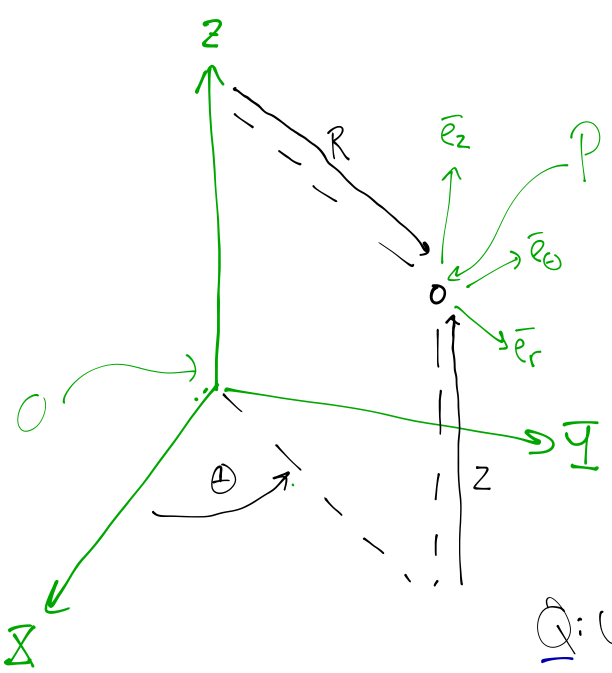
and

$$\bar{a}_P = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k} = \dot{v}_x\bar{i} + \dot{v}_y\bar{j} + \dot{v}_z k = a_x\bar{i} + a_y\bar{j} + a_z\bar{k}$$

Curvilinear Coordinates (Sec 2.3)

Common examples Spherical and Cylindrical, but also generalizable.

Cylindrical and Polar Coordinates (Sec. 2.3.1)



$$\bar{r}_{P/O} = R\bar{e}_r + z\bar{e}_z$$

Q: Why no $\bar{e}_\theta$?

need θ to know $\bar{e}_r$

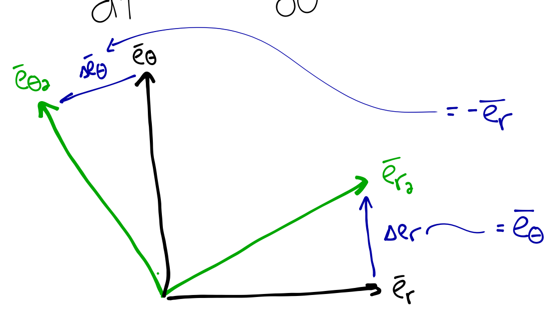
$$\bar{v}_P = \frac{d}{dt}\left(\bar{r}_{P/O}\right)$$

$$= \dot{R}\bar{e}_r + R\dot{\bar{e}}_r + \dot{z}\bar{e}_z + z\cancel{\dot{\bar{e}}_z}$$

Q: What is $\dot{\bar{e}}_r$?

$$\frac{d\bar{e}_r}{dt} = \dot{\theta}\frac{d\bar{e}_r}{d\theta}$$

$\bar{e}_r$ depends on θ, which depends on time



So, $\bar{v}_P = \dot{R}\bar{e}_r + R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z$

Cylindrical and Polar Coordinates (cont.)

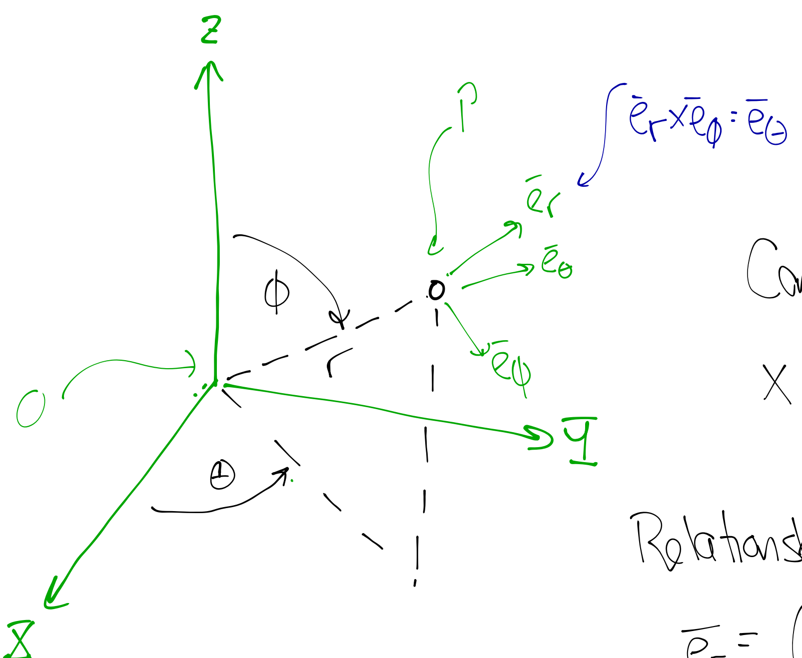
$$\bar{v}_P = \dot{R}\bar{e}_r + R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z$$

$$\bar{a}_P = \ddot{R}\bar{e}_r + \underbrace{\dot{R}\dot{\bar{e}}_r}_{\dot{\theta}\bar{e}_\theta} + \dot{R}\dot{\theta}\bar{e}_\theta + R\ddot{\theta}\bar{e}_\theta + \underbrace{R\dot{\theta}\dot{\bar{e}}_\theta}_{-\dot{R}\bar{e}_r} + \ddot{z}\bar{e}_z + \dot{z}\cancelto{0}{\dot{\bar{e}}_z}$$

$$\bar{a}_P = \left(\ddot{R} - R\dot{\theta}^2\right)\bar{e}_r + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\bar{e}_\theta + \ddot{z}\bar{e}_z$$

← We can, of course, use this set of coords to apply Newton's 2nd Law

Spherical Coordinates (Sec. 2.3.2)



limit $-\pi < \theta < \pi$

$0 \le \phi \le \pi$

Can define

$x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$

Relationship to Cartesian

$$\bar{e}_r = (\sin\phi\cos\theta)\bar{i} + (\sin\phi\sin\theta)\bar{j} + (\cos\phi)\bar{k}$$

$$\bar{e}_\phi = (\cos\phi\cos\theta)\bar{i} + (\cos\phi\sin\theta)\bar{j} - (\sin\phi)\bar{k}$$

$$\bar{e}_\theta = (-\sin\theta)\bar{i} + (\cos\theta)\bar{j}$$

$$\bar{r}_{P/O} = r\bar{e}_r$$

$$\bar{v}_P = \dot{r}\bar{e}_r + r\dot{\phi}\bar{e}_\phi + r\dot{\theta}\sin(\phi)\bar{e}_\theta$$

} see back for all the math on these

$$\bar{a}_P = \left(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi\right)\bar{e}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi\right)\bar{e}_\phi + \left(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi\right)\bar{e}_\theta$$

We can, of course, use this set of coords to apply Newton's 2nd Law

Mixed Kinematical Descriptions (Sec. 2.4)

Key point: It's best to use the coordinate system that most "naturally" fits to system or subsystem.

This may be different for different parts of the system, so

Q: How can we relate various coordinate systems?

Let's look at an example:

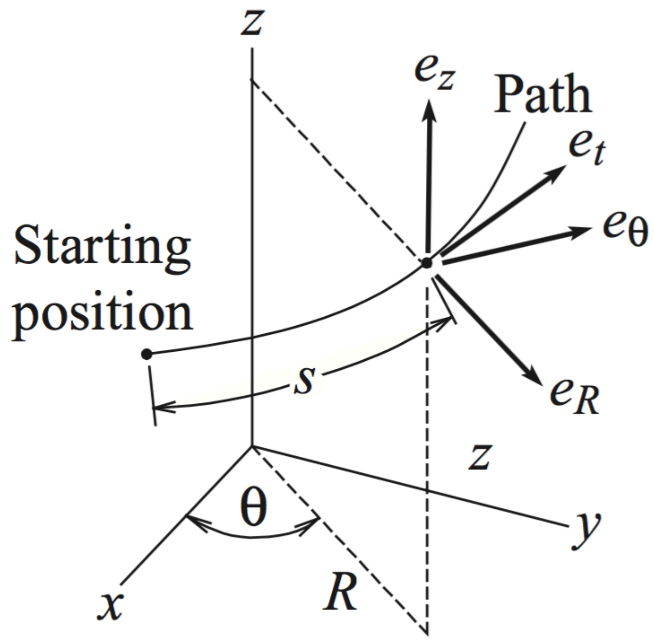


Figure 2.12. Mixed usage of path variables and cylindrical coordinates.

Depending on what we know about P, the path, and how those things are define cylindrical or path variables make sense here. To xchange between them, we need to know how unit vectors in one relate to the other.

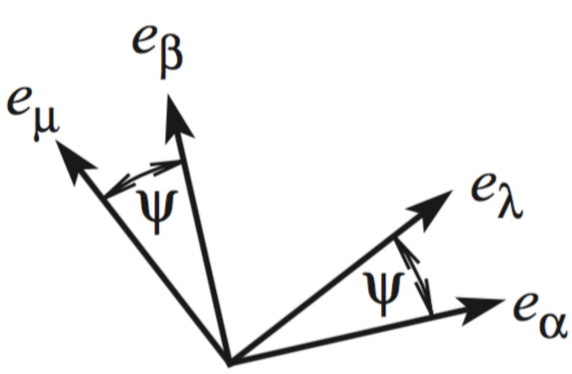


Figure 2.13. Relation between two sets of orthogonal unit vectors in a plane.

Let's look at a planar example:

$$\bar{e}_\lambda = \cos\psi\, \bar{e}_\alpha + \sin\psi\, \bar{e}_\beta$$

$$\bar{e}_\mu = -\sin\psi\, \bar{e}_\alpha + \cos\psi\, \bar{e}_\beta$$

Velocity of a point could be $\bar{v} = v_\alpha \bar{e}_\alpha + v_\beta \bar{e}_\beta = v_\lambda \bar{e}_\lambda + v_\mu \bar{e}_\mu$

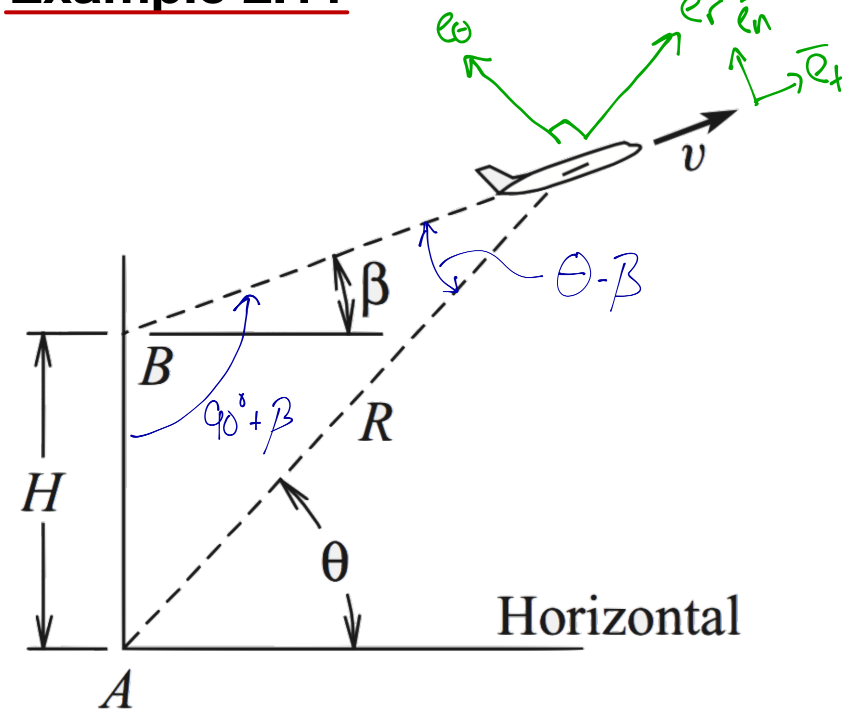
Given some velocity in $\bar{e}_\lambda, \bar{e}_\mu$ frame, what is velocity in $\bar{e}_\alpha, \bar{e}_\beta$ frame?

$$v_\alpha \bar{e}_\alpha + v_\beta \bar{e}_\beta = v_\lambda\left(\cos\psi\, \bar{e}_\alpha + \sin\psi\, \bar{e}_\beta\right) + v_\mu\left(-\sin\psi\, \bar{e}_\alpha + \cos\psi\, \bar{e}_\beta\right)$$

$$= \left[v_\lambda \cos\psi - v_\mu \sin\psi\right] \bar{e}_\alpha + \left[v_\lambda \sin\psi + v_\mu \cos\psi\right] \bar{e}_\beta$$

← equate components to find v_α and v_β

Example 2.11



Given $v \equiv$ const and $\beta \equiv$ const, what are $\dot{R}$ and $\dot{\theta}$?

$$\bar{v} = v\bar{e}_t = \dot{R}\bar{e}_r + R\dot{\theta}\bar{e}_\theta$$

$$\bar{v}\cdot\bar{e}_r = v\bar{e}_t\cdot\bar{e}_r = \dot{R} \quad \leftarrow \text{look at } \bar{e}_r \text{ direction}$$

$$\bar{e}_t = \cos(\theta-\beta)\bar{e}_r + \sin(\theta-\beta)\bar{e}_\theta$$

$$v\cos(\theta-\beta)\bar{e}_r + v\sin(\theta-\beta)\bar{e}_\theta = \dot{R}\bar{e}_r + R\dot{\theta}\bar{e}_\theta$$

Match vector components:

$$\boxed{v\cos(\theta-\beta) = \dot{R}} \quad \leftarrow \bar{e}_r \text{ direction}$$

$$v\sin(\theta-\beta) = R\dot{\theta} \quad \leftarrow \bar{e}_\theta \text{ direction}$$

$$\dot{\theta} = \frac{v}{R}\sin(\theta-\beta)$$

Q: What is R?

Use law of sines to find:

$$\frac{R}{\sin\left(\frac{\pi}{2}+\beta\right)} = \frac{H}{\sin(\theta-\beta)} \longrightarrow R = \frac{H\cos\beta}{\sin(\theta-\beta)} \quad \text{so,}$$

$$\boxed{\dot{\theta} = \frac{v}{H}\,\frac{\sin^2(\theta-\beta)}{\cos\beta}}$$

Check example 2.7 for solving this problem with a single coordinate system.