MCHE 513: Intermediate Dynamics

Fall 2016 – Mid-Term 2

Thursday, November 8

Name: Answer Key **ULID:** ___________

Directions: Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, using the paper provided to you. No calculators or equation sheets are allowed.

Academic Honesty (just a reminder):

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

______ **Problem 1 – 30 Points**

The sketch in Figure 1 shows a uniform semicylinder of mass m and radius R on a surface with friction μ. Gravity acts on the system.

For this system:

a. Write the equations of motion.

b. What is the minimum coefficient of friction, μ, that is necessary to ensure no slipping for a given angular velocity, $\dot{\theta}$?

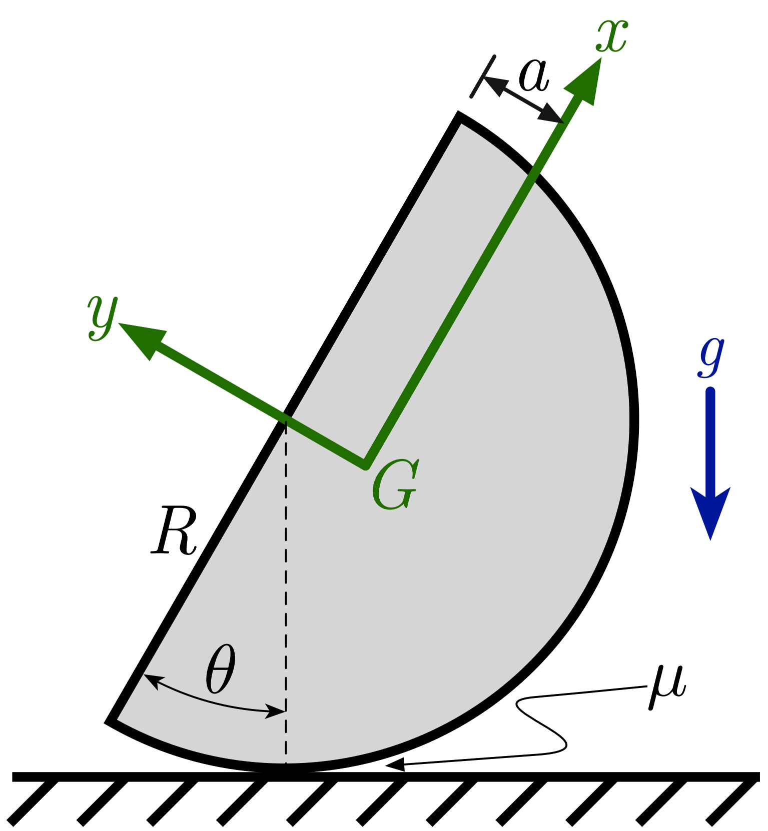
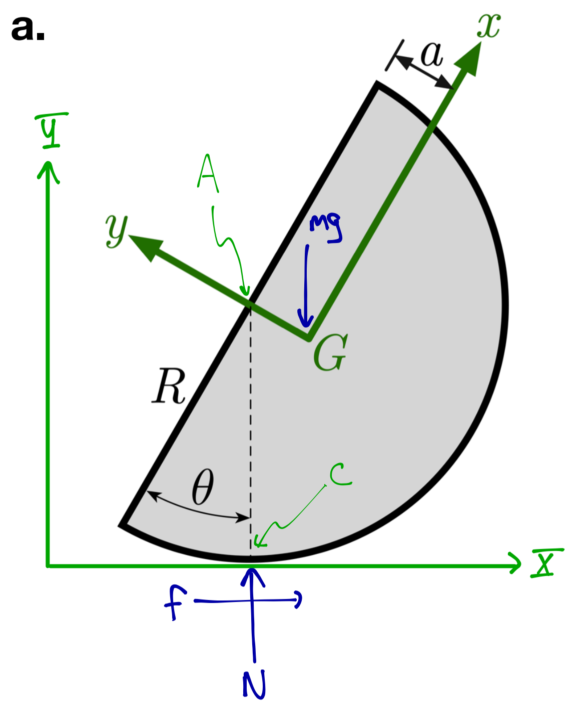


Figure 1: A Uniform Semicylinder

Problem 1 (cont.)

a.



We'll assume no slipping here, then work out what μ must be to ensure that.

If no slipping we can relate the angular velocity to the velocity of point A

If no slipping,

$$v_A = R\dot{\theta}\,\bar{I} \quad \text{and} \quad \bar{a}_A = R\ddot{\theta}$$

The angular velocity of frame xyz are

$$\bar{\omega}_{xyz} = -\dot{\theta}\,\bar{k} = -\dot{\theta}\,\bar{K} \quad \text{and} \quad \bar{\alpha}_{xyz} = -\ddot{\theta}\,\bar{k} = -\ddot{\theta}\,\bar{K}$$

rotation from XYZ to xyz is $(90-\theta)$

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} \cos(90-\theta) & +\sin(90-\theta) & 0 \\ -\sin(90-\theta) & \cos(90-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix}$$

$$\bar{r}_{G/A} = -a\bar{j} \longrightarrow \bar{r}_{G/A} = -a\left(-\cos\theta\,\bar{I} + \sin\theta\,\bar{J}\right) = a\cos\theta\,\bar{I} - a\sin\theta\,\bar{J}$$

$$\bar{a}_G = \bar{a}_A + \bar{\alpha}_{xyz} \times \bar{r}_{G/A} + \bar{\omega}_{xyz} \times \left(\bar{\omega}_{xyz} \times \bar{r}_{G/A}\right)$$

$$= R\ddot{\theta}\,\bar{I} + \left[-\ddot{\theta}\,\bar{K} \times \left(a\cos\theta\,\bar{I} - a\sin\theta\,\bar{J}\right)\right] + \left[-\dot{\theta}\,\bar{K} \times \left(\dot{\theta}\,\bar{K} \times \left(a\cos\theta\,\bar{I} - a\sin\theta\,\bar{J}\right)\right)\right]$$

$$= R\ddot{\theta}\,\bar{I} + \left[-a\ddot{\theta}\cos\theta\,\bar{J} - a\ddot{\theta}\sin\theta\,\bar{I}\right] + \left[-a\dot{\theta}^2\cos\theta\,\bar{I} + a\dot{\theta}^2\sin\theta\,\bar{J}\right]$$

$$\bar{a}_G = \left[R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right]\bar{I} + \left[-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta\right]\bar{J}$$

$$\Sigma\bar{F} = f\,\bar{I} + (N - mg)\,\bar{J}$$

Problem 1 (cont.)

$$f\bar{I} + (N - mg)\bar{J} = m\left[\left[R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right]\bar{I} + \left[-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta\right]\bar{J}\right]$$

We can write this in the $\bar{I}$ and $\bar{J}$ directions

$$\bar{I} \rightarrow m\left(R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right) = f$$ ← could also explicitly write $f = \mu N$

$$\bar{J} \rightarrow m\left(-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta\right) = N - mg$$

For rotation, sum moments about G

$$\sum \bar{M}_G = \bar{r}_{c/G} \times \left(f\bar{I} + N\bar{J}\right)$$

$$\bar{r}_{c/G} = a\bar{j} - R\bar{J}$$

$$= a\left(-\cos\theta\bar{I} + \sin\theta\bar{J}\right) - R\bar{J}$$

$$\bar{r}_{c/G} = -a\cos\theta\bar{I} + \left(-R + a\sin\theta\right)\bar{J}$$

So,

$$\sum \bar{M}_G = \left[-a\cos\theta\bar{I} + \left(-R + a\sin\theta\right)\bar{J}\right] \times \left[f\bar{I} + N\bar{J}\right]$$

$$= \left[-aN\cos\theta + f\left(R - a\sin\theta\right)\right]\bar{K}$$

$$I_{zz}\ddot{\theta} = -aN\cos\theta + f\left(R - a\sin\theta\right)$$ ← all in K direction

These three equations (boxed in orange) represent the eq. of motion for a 1DOF system. To reduce to a single eq. of motion, solve the linear equations for f and N, then substitute into the moment equation.

$$I_{zz}\ddot{\theta} = \left(-a\cos\theta\right)\left(m\left(-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta + g\right)\right)$$
$$+ \left(m\left(R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right)\right)\left(R - a\sin\theta\right)$$

Problem 1 (cont.)

b. We have 3 equations that are based on assuming no slip. They must be consistent.

$$m\left(R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right) = f$$

$$m\left(-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta\right) = N - mg \longrightarrow N = m\left(-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta + g\right)$$

$$I_{zz}\ddot{\theta} = -aN\cos\theta + f\left(R - a\sin\theta\right)$$

We know that $f \propto \mu N$, so $f \propto \mu\left[m\left(-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta + g\right)\right]$

Substitute N and f into the rotation equation and solve for μ

Alternately, we could write $f = \mu N$ as

$$\mu = \frac{f}{N} = \frac{\cancel{m}\left(R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right)}{\cancel{m}\left(-a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta + g\right)}$$

______ **Problem 2 – 40 Points**

The system in Figure 2 consists of bar AB of mass m and length L, which slides in the frictionless collar at point C. The collar is connected to a vertical shaft by a perfect, frictionless pin. The vertical bar rotates at a constant rate, $\dot{\psi}$. The angle of the bar from vertical is described by θ, and the displacement of its center of mass, G, from the pin, C, is described by x.

a. Write the velocity of the bar's center of mass.

b. Write the acceleration of the bar's center of mass.

c. Write the angular velocity of the bar.

d. Write the equations of motion governing the motion of x and θ.

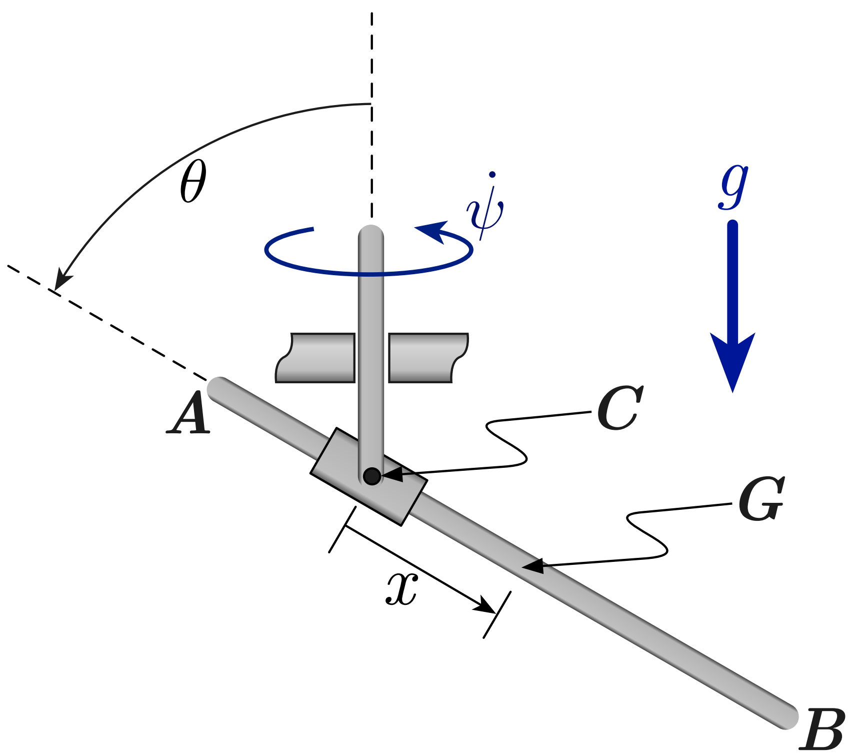
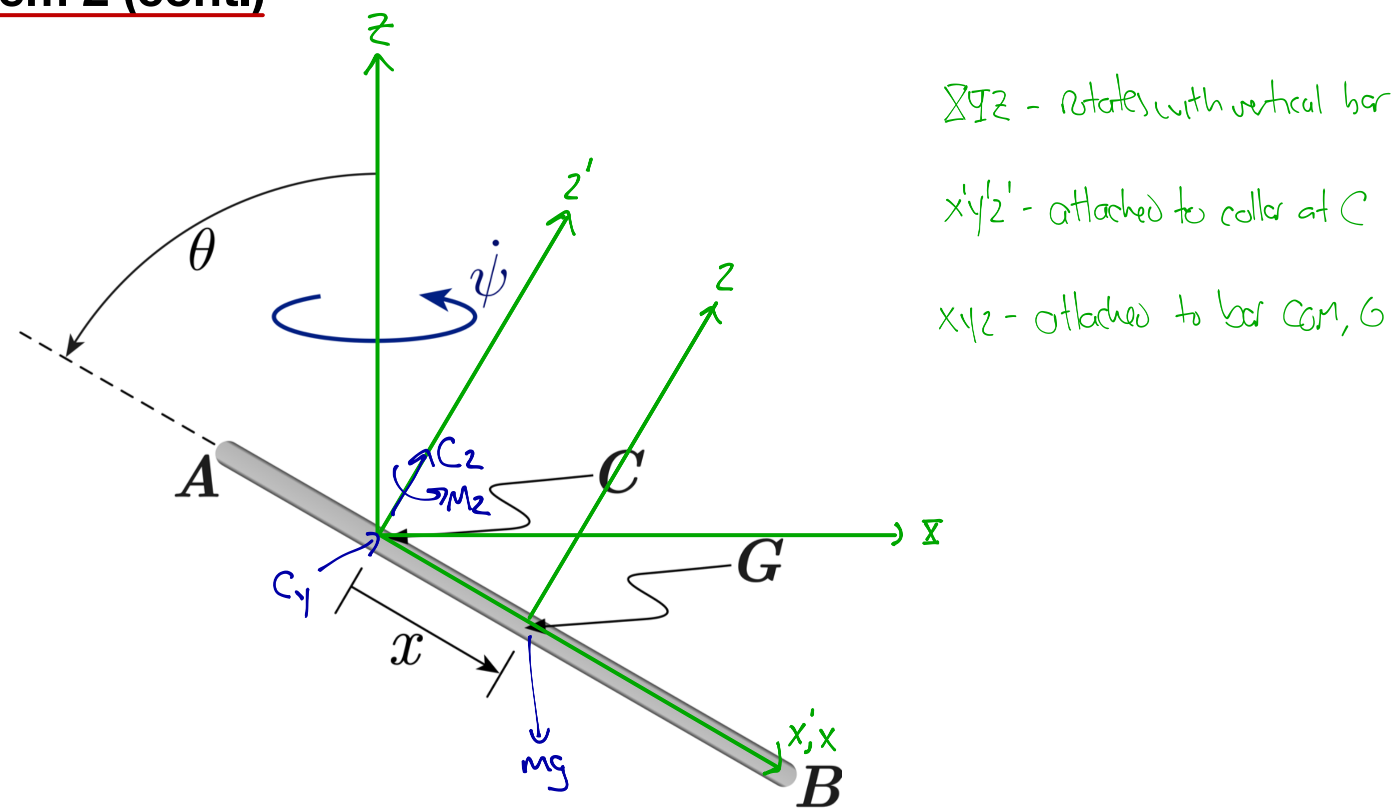


Figure 2: Disk Being Pulled

Problem 2 (cont.)

a.



$$\bar{v}_G = \bar{v}_C^{\,0} + (\bar{v}_G)_{x'y'z'} + \bar{\omega}_{x'y'z'} \times \bar{r}_{G/C}$$

$$(\bar{v}_G)_{x'y'z'} = \dot{x}\,\bar{i} \qquad \bar{r}_{G/A} = x\,\bar{i}$$

$$\bar{\omega}_{x'y'z'} = \dot{\psi}\,\bar{K} - \dot{\theta}\,\bar{j}$$
$$= \dot{\psi}(-\cos\theta\,\bar{i} + \sin\theta\,\bar{k}) - \dot{\theta}\,\bar{j}$$

$$= \dot{x}\,\bar{i} + (-\dot{\psi}\cos\theta\,\bar{i} - \dot{\theta}\,\bar{j} + \dot{\psi}\sin\theta\,k) \times (x\,\bar{i})$$

$$= \dot{x}\,\bar{i} + (x\dot{\theta}\,\bar{k} + x\dot{\psi}\sin\theta\,\bar{j})$$

$$\boxed{\bar{v}_G = \dot{x}\,\bar{i} + x\dot{\psi}\sin\theta\,\bar{j} + x\dot{\theta}\,\bar{k}}$$

b. $\bar{a}_G = \bar{a}_C^{\,0} + (\bar{a}_G)_{x'y'z'} + \bar{\alpha}_{x'y'z'} \times \bar{r}_{G/A} + (v_G)_{x'y'z'} + \bar{\omega}_{x'y'z'} \times (\bar{\omega}_{x'y'z'} \times \bar{r}_{G/A}) \times 2\omega_{x'y'z'} \times (\bar{v}_G)_{x'y'z'}$

$$\bar{\alpha}_{x'y'z'} = \frac{d\omega_{x'y'z'}}{dt} = \ddot{\psi}\bar{K}^{\,0} - \ddot{\theta}\,\bar{j} - \dot{\theta}(\omega_{x'y'z'} \times \bar{j})$$

$$= -\ddot{\theta}\,\bar{j} - \dot{\theta}(-\dot{\psi}\cos\theta\,\bar{i} + \dot{\psi}\sin\theta\,\bar{k} \times \bar{j}) = -\ddot{\theta}\,\bar{j} - \dot{\theta}\dot{\psi}\cos\theta\,\bar{k} + \dot{\theta}\dot{\psi}\sin\theta\,\bar{i}$$

$$(a_G)_{x'y'z'} = \ddot{x}\,\bar{i}$$

Problem 2 (cont.)

$$\bar{\alpha}_{x'y'z'} \times \bar{r}_{G/A} = \left(\dot{\theta}\dot{\psi}\sin\theta\,\bar{i} - \ddot{\theta}\,\bar{j} + \dot{\theta}\dot{\psi}\cos\theta\,\bar{k}\right) \times x\,\bar{i}$$

$$= +x\ddot{\theta}\,\bar{k} + x\dot{\theta}\dot{\psi}\cos\theta\,\bar{j}$$

$$\omega_{x'y'z'} \times \left(\omega_{x'y'z'} \times \bar{r}_{G/A}\right) = \omega_{x'y'z'} \times \left(x\dot{\theta}\,\bar{k} + x\dot{\psi}\sin\theta\,\bar{j}\right)$$

$$= \left(-\dot{\psi}\cos\theta\,\bar{i} - \dot{\theta}\,\bar{j} + \dot{\psi}\sin\theta\,\bar{k}\right) \times \left(x\dot{\theta}\,\bar{k} + x\dot{\psi}\sin\theta\,\bar{j}\right)$$

$$= x\dot{\psi}\dot{\theta}\cos\theta\,\bar{j} - x\dot{\psi}^2\cos\theta\sin\theta\,k - x\ddot{\theta}\,\bar{i} - x\dot{\psi}^2\sin^2\theta\,\bar{i}$$

$$2\omega_{x'y'z'} \times \left(\bar{v}_G\right)_{x'y'z'} = 2\left(-\dot{\psi}\cos\theta\,\bar{i} - \dot{\theta}\,\bar{j} + \dot{\psi}\sin\theta\,k\right) \times \dot{x}\,\bar{i}$$

$$= 2\left[\dot{x}\dot{\theta}\,\bar{k} + \dot{x}\dot{\psi}\sin\theta\,\bar{j}\right]$$

So

$$\bar{a}_G = \left[\ddot{x}\,\bar{i}\right] + \left[-x\dot{\theta}\dot{\psi}\cos\theta\,\bar{j} - x\ddot{\theta}\,\bar{k}\right] + \left[-x\ddot{\theta}\,\bar{i} - x\dot{\psi}^2\sin^2\theta\,\bar{i} + x\dot{\psi}\dot{\theta}\cos\theta\,\bar{j} - x\dot{\psi}^2\cos\theta\sin\theta\,\bar{k}\right]$$

$$+ 2\dot{x}\dot{\psi}\sin\theta\,\bar{j}$$

$$\bar{a}_G = \left[\ddot{x} - x\ddot{\theta} - x\dot{\psi}^2\sin^2\theta\right]\bar{i} + \left[+x\dot{\theta}\dot{\psi}\cos\theta + x\dot{\psi}\dot{\theta}\cos\theta + 2\dot{x}\dot{\psi}\sin\theta\right]\bar{j}$$
$$+ \left[+x\ddot{\theta} + 2\dot{x}\dot{\theta} - x\dot{\psi}^2\cos\theta\sin\theta\right]\bar{k}$$

c. $\bar{\omega}_{bar} = \dot{\psi}\,\bar{K} - \dot{\theta}\,\bar{J} = \dot{\psi}\left(-\cos\theta\,\bar{i} + \sin\theta\,\bar{k}\right) - \dot{\theta}\,\bar{j}$

$$\bar{\omega}_{bar} = -\dot{\psi}\cos\theta\,\bar{i} - \dot{\theta}\,\bar{j} + \dot{\psi}\sin\theta\,\bar{k}$$

$$\bar{\alpha}_{bar} = \left(\dot{\theta}\dot{\psi}\sin\theta\,\bar{i} - \ddot{\theta}\,\bar{j} + \dot{\theta}\dot{\psi}\cos\theta\,\bar{k}\right)$$

Problem 2 (cont.)

d.

$$\bar{a}_G = \left[\ddot{x} - x\dot{\theta}^2 - x\dot{\psi}^2\sin^2\theta\right]\bar{i} + \left[+x\ddot{\theta}\dot{\psi}\cos\theta + x\dot{\psi}\dot{\theta}\cos\theta + 2\dot{x}\dot{\psi}\sin\theta\right]\bar{j}$$
$$+\left[+x\ddot{\theta} + 2\dot{x}\dot{\theta} - x\dot{\psi}^2\cos\theta\sin\theta\right]\bar{k}$$

$$\Sigma\bar{F} = -mg\bar{I} + C_y\bar{j} + C_z\bar{k} = -mg\left(-\cos\theta\,\bar{i} + \sin\theta\,\bar{k}\right) + C_y\bar{j} + C_z\bar{k}$$
$$= mg\cos\theta\,\bar{i} + C_y\bar{j} + \left(C_z - mg\sin\theta\right)\bar{k}$$

Write the equation for each direction

$$\bar{i} \rightarrow \quad m\left[\ddot{x} - x\dot{\theta}^2 - x\dot{\psi}^2\sin^2\theta\right] = mg\cos\theta$$
$$\bar{j} \rightarrow \quad m\left[+x\dot{\theta}\dot{\psi}\cos\theta + x\dot{\psi}\dot{\theta}\cos\theta + 2\dot{x}\dot{\psi}\sin\theta\right] = C_y$$
$$\bar{k} \rightarrow \quad m\left[+x\ddot{\theta} + 2\dot{x}\dot{\theta} - x\dot{\psi}^2\cos\theta\sin\theta\right] = C_z - mg\sin\theta$$

We also need to write the moment equations

$$\Sigma\bar{M}_G = \bar{r}_{c/G} \times \left(C_y\bar{j} + C_z\bar{k}\right) + M_z\bar{k} \qquad \bar{r}_{o/G} = -x\bar{i}$$
$$= -x\bar{i} \times \left(C_y\bar{j} + C_z\bar{k}\right) + M_z\bar{k}$$
$$\Sigma\bar{M}_G = \left(-xC_y + M_z\right)\bar{k} + xC_z\bar{j}$$

For this bar $I_{xx} = 0$ and $I_{yy} = I_{zz}$ axis xyz is principle, so we can use Euler's Eq.

$$\Sigma\bar{M}_G \cdot \bar{i} = 0$$
$$\Sigma\bar{M}_G \cdot \bar{j} = xC_z = I_{yy}\alpha_y - \left(I_{zz} - I_{xx}\right)\omega_x\omega_z \quad (I_{xx} \to 0)$$
$$\Sigma M_G \cdot \bar{k} = -xC_y + M_z = I_{zz}\alpha_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y \quad (I_{xx} \to 0)$$

$\omega_p = p^{th}$ direction of angular vel.
$\alpha_p = p^{th}$ direction of ang accel.

Problem 2 (cont.)

The $\bar{\imath}$ direction linear equation is the eq. of motion for x

To fully define the equation of motion for θ, we need to

1) so $\bar{k}$ direction linear equation for C_2, then

2) substitute into the $\bar{J}$-direction moment equation

To find M_2,

1) solve $\bar{J}$-direction linear equation for C_1, then

2) substitute into the $\bar{k}$-direction moment equation

______ **Problem 3 – 30 Points**

Figure 3 shows two links connected via a frictionless pin at point A. The first link has mass m_1 and length L_1. It is connected to a grounded, frictionless pin at point O at height h above the lower surface. Its angle relative to vertical is described by θ. The lower link's mass and length are described by m_2 and L_2, respectively. The lower link also contacts a frictionless surface at point B, where an always-horizontal force, F, is also applied. Gravity acts on the total system.

a. Write the equations of motion for this system.

b. What must the force, F, be in order to hold the system at a static, nonzero equilibrium angle, θ_0, where $0 < \theta_0 < 90°$? In other words, what must F be to hold the system in the configuration shown in Figure 3?

c. What is the virtual displacement of point B, $\delta\bar{r}_b$?

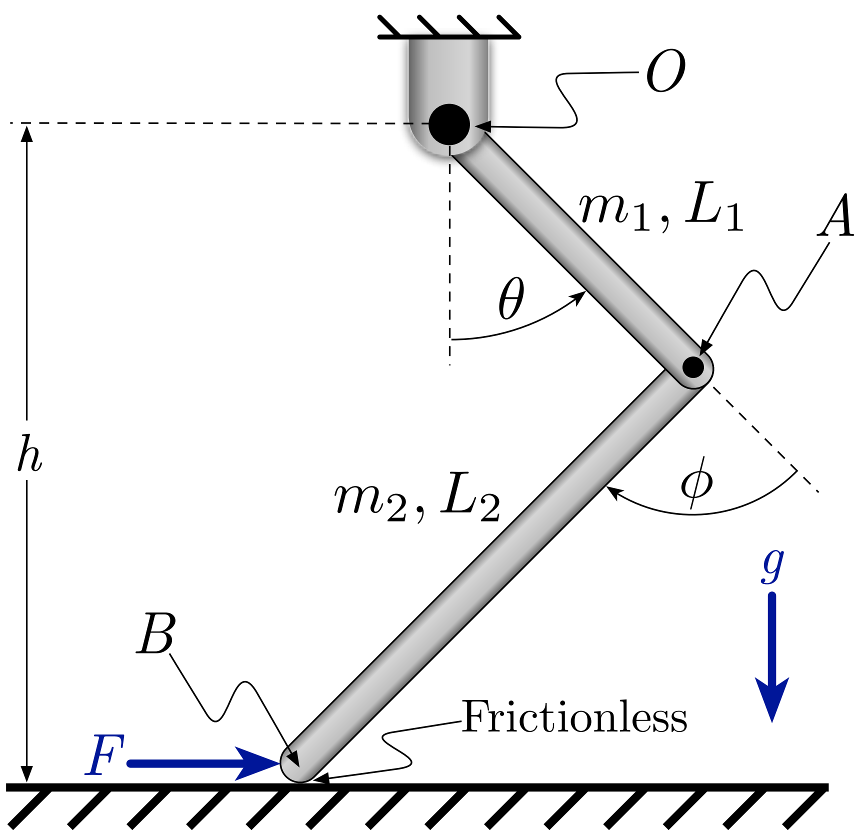
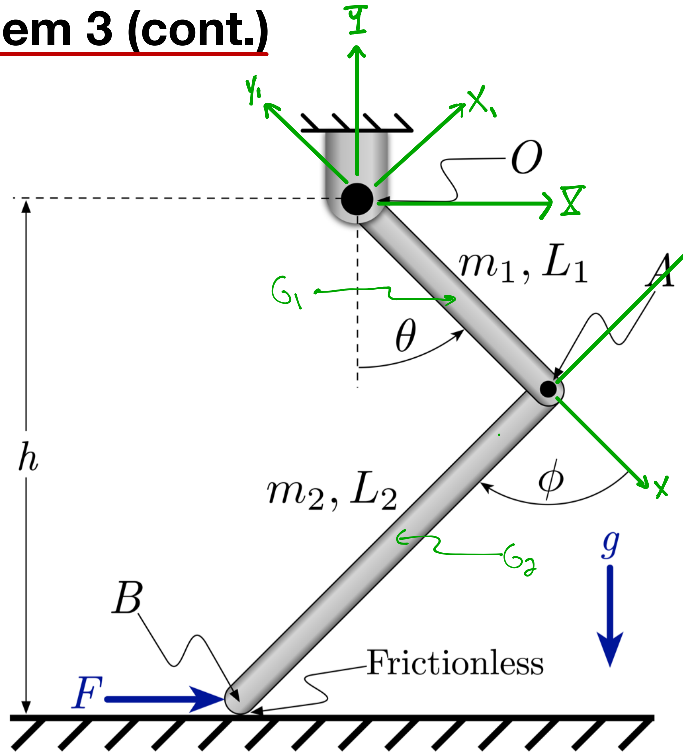
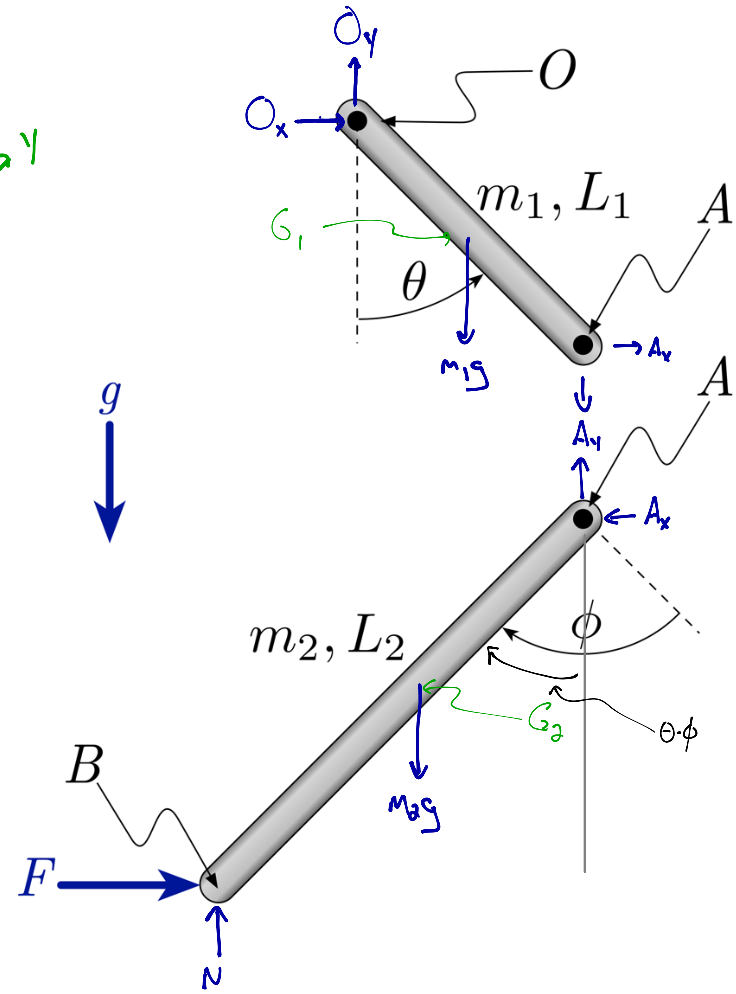


Figure 3: A Two-Link System

Problem 3 (cont.)

a.





XYZ - fixed, origin at O

$x_1 y_1 z_1$ - fixed to link OA

xyz - fixed to link AB

Use θ and ϕ to write the eq. of motion

For bar OA, we can sum moments about point O. It's in pure rotation about that point

For bar AB, we'll need to use the COM.

$$\Sigma \bar{M}_O = I_O \ddot{\theta} \bar{K}$$ where I_O is the I_{zz} moment of inertia of link OA about point O

$$\Sigma \bar{M}_O = \left(\bar{r}_{G_1/O} \times -mg\bar{J}\right) + \left(\bar{r}_{A/O} \times \left(A_x\bar{I} - A_y\bar{J}\right)\right)$$

$$\bar{r}_{G_1/O} = \frac{L_1}{2}\sin\theta\,\bar{I} - \frac{L_1}{2}\cos\theta\,\bar{J} \qquad \bar{r}_{A/O} = L_1\sin\theta\,\bar{I} - L_1\cos\theta\,\bar{J}$$

$$\Sigma \bar{M}_O = \left(-mg\frac{L_1}{2}\sin\theta\,\bar{K}\right) + \left(-A_y L_1\cos\theta\,\bar{K} + A_x L_1 \sin\theta\,\bar{K}\right)$$ ← all in $\bar{K}$, as it should be

$$I_O\ddot{\theta} = A_x L_1\cos\theta - A_y L_1 \sin\theta - mg\frac{L_1}{2}\sin\theta$$

Problem 3 (cont.)

$\sum \bar{M}_{G_2} = I_{G_2}(\ddot{\theta} - \ddot{\phi})\bar{K}$ where I_{G_2} is the I_{zz} moment of inertia of link AB about its COM

$$\sum \bar{M}_{G_2} = \left(\bar{r}_{A/G_2} \times \left(-A_x\bar{I} + A_y\bar{J}\right)\right) + \left(\bar{r}_{B/G_2} \times \left(F\bar{I} + N\bar{J}\right)\right)$$

$$\bar{r}_{A/G_2} = \frac{L_2}{2}\sin(\theta-\phi)\bar{I} + \frac{L_2}{2}\cos(\theta-\phi)\bar{J} \qquad \bar{r}_{B/G_2} = -\frac{L_2}{2}\sin(\theta-\phi)\bar{I} - \frac{L_2}{2}\cos(\theta-\phi)\bar{J}$$

$$\sum \bar{M}_{G_2} = \left(A_y\frac{L_2}{2}\sin(\theta-\phi)\bar{K} + A_x\frac{L_2}{2}\cos(\theta-\phi)\bar{K}\right) + \left(-N\frac{L_2}{2}\sin(\theta-\phi)\bar{K} + F\frac{L_2}{2}\cos(\theta-\phi)\bar{K}\right) = I_{G_2}(\ddot{\theta}-\ddot{\phi})\bar{K}$$

All in $\bar{K}$, as it should be

For link AB, we also need to write the "F=ma" equations. They will also help us solve for reactions A_x, A_y, and N

$$\sum \bar{F} = m_2\bar{a}_{G_2}$$

$$\bar{\omega} = -\dot{\phi}\bar{K} = -\dot{\phi}\bar{k}_1 = \dot{\phi}k$$

$$\bar{\alpha} = -\ddot{\phi}\bar{K} = \ddot{\phi}\bar{k}_1 = \ddot{\phi}\bar{k}$$

$$\bar{a}_{G_2} = \bar{a}_A + \bar{\alpha}\times\bar{r}_{G_2/A} + \bar{\omega}\times\bar{\omega}\times\bar{r}_{G_2/A}$$

point A is in pure rotation about O so we can write

$$\bar{a}_A = L_1\ddot{\theta}\bar{i}_1 + L_1\dot{\theta}^2\bar{j}_1$$

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{bmatrix}$$

$$\bar{r}_{G_2/A} = \frac{-L_2}{2}\bar{J} = \frac{-L_2}{2}\sin\phi\,\bar{i}_1 + \frac{L_2}{2}\cos\phi\,\bar{j}_1$$

$$\bar{a}_{G_2} = L_1\ddot{\theta}\bar{i}_1 + L_1\dot{\theta}^2\bar{j}_1 + \left(-\ddot{\phi}\bar{k}_1 \times \left(\frac{-L_2}{2}\sin\phi_1\bar{i}_1 + \frac{L_2}{2}\cos\phi\,\bar{j}_1\right)\right) + \dot{\phi}\bar{k}_1 \times \left(\dot{\phi}\bar{k}_1 \times \left(-\frac{L_2}{2}\sin\phi_1\bar{i}_1 + \frac{L_2}{2}\cos\phi\,\bar{j}_1\right)\right)$$

$$\sum \bar{F} = (F - A_x)\bar{I} + (N + A_y)\bar{J} = m\bar{a}_G$$

The three orange boxes here provide all the information necessary to form the eq. of motion. They also provide the necessary information to eliminate the reaction forces from these eq.

Problem 3 (cont.)

b. Using our solution from part A, we see that

$$\sum \bar{F} = (F - A_x)\bar{I} + (N + A_y)\bar{J} = m\bar{a}_G$$

If held stationary/static $\bar{a}_G = 0$ so

$$\sum \bar{F} = (F - A_x)\bar{I} + (N + A_y)\bar{J} = 0 \rightarrow$$

$F - A_x = 0$ and $N + A_y = 0$

↑

The applied force must balance the reaction in the I direction at A

We can use the solution of these reactions from part a to find F in terms of θ and ϕ →

c. To write $\delta \bar{r}_B$, let's use what the book calls the kinematical method. 1st with the velocity of point B

$$\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A}$$

A is in pure rotation about point O so $\bar{v}_A = L_1 \dot{\theta} \bar{i}_1$

It's easiest to write $\bar{r}_{B/A}$ in the xyz frame, $\bar{r}_{B/A} = -L_2 \bar{J}$

Then $\bar{\omega}_{xyz} = -\dot{\phi} k$

$$\begin{bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix}$$

$$\bar{v}_B = L_1 \dot{\theta} \bar{i}_1 + \left(-\dot{\phi}\bar{k} \times -L_2 \bar{J}\right)$$

$$= L_1 \dot{\theta}\left(\cos\phi\, \bar{i} + \sin\phi\, \bar{j}\right) + \left(-L_2 \dot{\phi}\, \bar{i}\right)$$

$$\bar{v}_B = \left(L_1 \cos\phi\, \dot{\theta} - L_2 \dot{\phi}\right)\bar{i} + \left(L_1 \sin\phi\, \dot{\theta}\right)\bar{j}$$

Now replace $\dot{\theta}$ with $\delta\theta$ and $\dot{\phi}$ with $\delta\phi$ to map $v_B \rightarrow \delta \bar{r}_B$

$$\boxed{\delta \bar{r}_B = \left(L_1 \cos\phi\, \delta\theta - L_2\, \delta\phi\right)\bar{i} + \left(L_1 \sin\phi\, \delta\theta\right)\bar{j}}$$