MCHE 513: Intermediate Dynamics

Fall 2018 – Mid-Term 1

Tuesday, October 9

Name: Answer Key ______________________ **ULID:** __________

Directions: Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, using the paper provided to you. No calculators or equation sheets are allowed.

Academic Honesty (just a reminder):

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

______ Problem 1 – 40 Points

In Figure 1, a disk of radius R spins about link BC at a rate of $\dot{\phi}$. Link ABC rotates about axis Z at a constant rate, $\dot{\psi}$. The angle from link AB from vertical, θ, is also variable. The angle of the bend in the ABC linkage, β, is fixed.

a. Write the angular velocity and angular acceleration of the disk. For each, be sure to indicate how to resolve all the components into the same frame.

b. What is the velocity of point D?

c. What is the acceleration of point D?

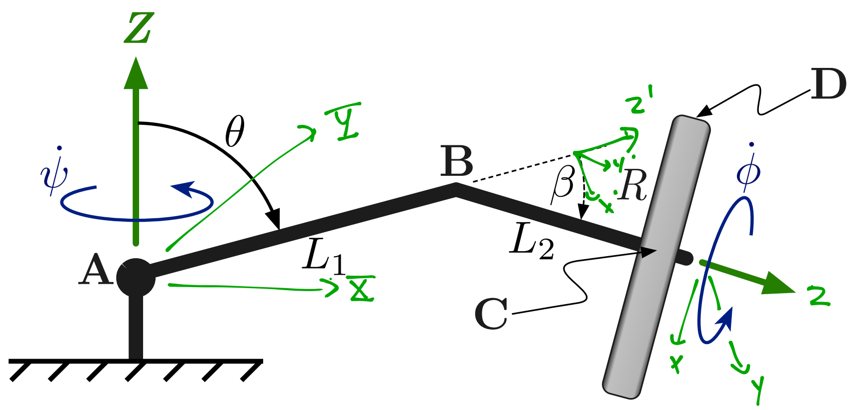


Figure 1: A Spinning disk on a Bent Linkage

a. Define coord. frames

XYZ - fixed to AB, but X always horizontal

x'y'z' - fixed to AB, with z' along AB

xyz - fixed to BC, with z along BC

$$\begin{bmatrix} \bar{i}' \\ \bar{j}' \\ k' \end{bmatrix} = R_\theta \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{bmatrix} \rightarrow R_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \bar{i} \\ \bar{j} \\ k \end{bmatrix} = R_\beta \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ k' \end{bmatrix} = \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} \bar{i}' \\ \bar{j}' \\ k' \end{bmatrix} \rightarrow R_\beta = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

Problem 1 (cont.)

a.

$$R_\beta R_\theta = \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} = \begin{bmatrix} c\beta c\theta - s\beta s\theta & 0 & -c\beta s\theta - s\beta c\theta \\ 0 & 1 & 0 \\ s\beta c\theta + c\beta s\theta & 0 & -s\beta s\theta + c\beta c\theta \end{bmatrix}$$

so $\bar{k} = \left(s\beta c\theta + c\beta s\theta\right)\bar{I} + \left(-s\beta s\theta + c\beta c\theta\right)\bar{K}$

$$\bar{\omega}_{disk} = \dot{\psi}\bar{K} + \dot{\theta}\bar{J} + \dot{\phi}\bar{k} = \dot{\psi}\bar{K} + \dot{\theta}\bar{J} + R_\beta R_\theta \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$\boxed{\bar{\omega}_{disk} = \dot{\phi}\left(s\beta c\theta + c\beta s\theta\right)\bar{I} + \dot{\theta}\bar{J} + \left[\dot{\phi}\left(-s\beta s\theta + c\beta c\theta\right) + \dot{\psi}\right]\bar{K}}$$

$$\bar{\alpha}_{disk} = \cancelto{0}{\ddot{\psi}}\bar{K} + \dot{\psi}\cancelto{0}{\dot{\bar{K}}} + \ddot{\theta}\bar{J} + \dot{\theta}\dot{\bar{J}} + \ddot{\phi}\bar{k} + \dot{\phi}\dot{\bar{k}}$$

$$= \ddot{\theta}\bar{J} + \dot{\theta}\left(\bar{\omega}_{XYZ} \times \bar{J}\right) + \ddot{\phi}\bar{k} + \dot{\phi}\left(\bar{\omega}_{xyz} \times \bar{k}\right)$$

where $\bar{\omega}_{XYZ}$ is angular velocity of XYZ frame – $\bar{\omega}_{XYZ} = \dot{\psi}\bar{K}$

ω_{xyz} is angular velocity of xyz frame – $\bar{\omega}_{xyz} = \dot{\psi}\bar{K} + \dot{\theta}\bar{J}$

$$\bar{\alpha}_{disk} = \ddot{\theta}\bar{J} + \dot{\theta}\left(\dot{\psi}\bar{K} \times \bar{J}\right) + \ddot{\phi}\bar{k} + \dot{\phi}\left[\left(\dot{\psi}\bar{K} + \dot{\theta}\bar{J}\right) \times \bar{k}\right]$$

$$\boxed{\bar{\alpha}_{disk} = \ddot{\theta}\bar{J} + \left(-\dot{\theta}\dot{\psi}\bar{I}\right) + \ddot{\phi}\bar{k} + \dot{\phi}\left[\left(\dot{\psi}\bar{K} + \dot{\theta}\bar{J}\right) \times \bar{k}\right]}$$

← see above for $\bar{k}$ in terms of $\bar{I}$ and $\bar{K}$ to allow this cross product

See the Jupyter Notebook for the completed expression

Problem 1 (cont.)

b. Treat point D as the "top" of the disk as it is in this instant

$$\bar{v}_D = \bar{v}_C + \bar{\omega}_{disk} \times \bar{r}_{D/C} \qquad - \quad \bar{r}_{D/C} = -R\bar{i} \quad , \ \bar{\omega}_{disk} \text{ defined in part a)}$$

$$\bar{v}_C = \cancel{\bar{v}_A}^{\,0} + \bar{\omega}_{xyz} \times \bar{r}_{C/A}$$

$$\bar{r}_{C/A} = L_1 \bar{k}' + L_2 \bar{k} \qquad \bar{\omega}_{xyz} = \dot{\psi}\bar{K} + \dot{\theta}\bar{J}$$

$$= L_1\left(\sin\theta \bar{I} + \cos\theta \bar{K}\right) + L_2\left[\left(s\beta c\theta + c\beta s\theta\right)\bar{I} + \left(-s\beta s\theta + c\beta c\theta\right)\bar{K}\right]$$

c.

$$\bar{a}_D = \bar{a}_C + \left(\bar{\alpha}_{disk} \times \bar{r}_{D/C}\right) + \bar{\omega}_{disk} \times \left(\bar{\omega}_{disk} \times \bar{r}_{D/C}\right)$$

$$\bar{a}_C = \cancel{\bar{a}_A}^{\,0} + \left(\bar{\alpha}_{xyz} \times \bar{r}_{C/A}\right) + \bar{\omega}_{xyz} \times \left(\bar{\omega}_{xyz} \times \bar{r}_{C/A}\right)$$

$$\bar{\alpha}_{xyz} = \ddot{\theta}\bar{J} + \dot{\theta}\dot{\bar{J}} = \ddot{\theta}\bar{J} + \dot{\theta}\left(\dot{\psi}\bar{K} \times \bar{J}\right) = -\dot{\theta}\dot{\psi}\bar{I} + \ddot{\theta}\bar{J}$$

all other quantities defined earlier in the problem

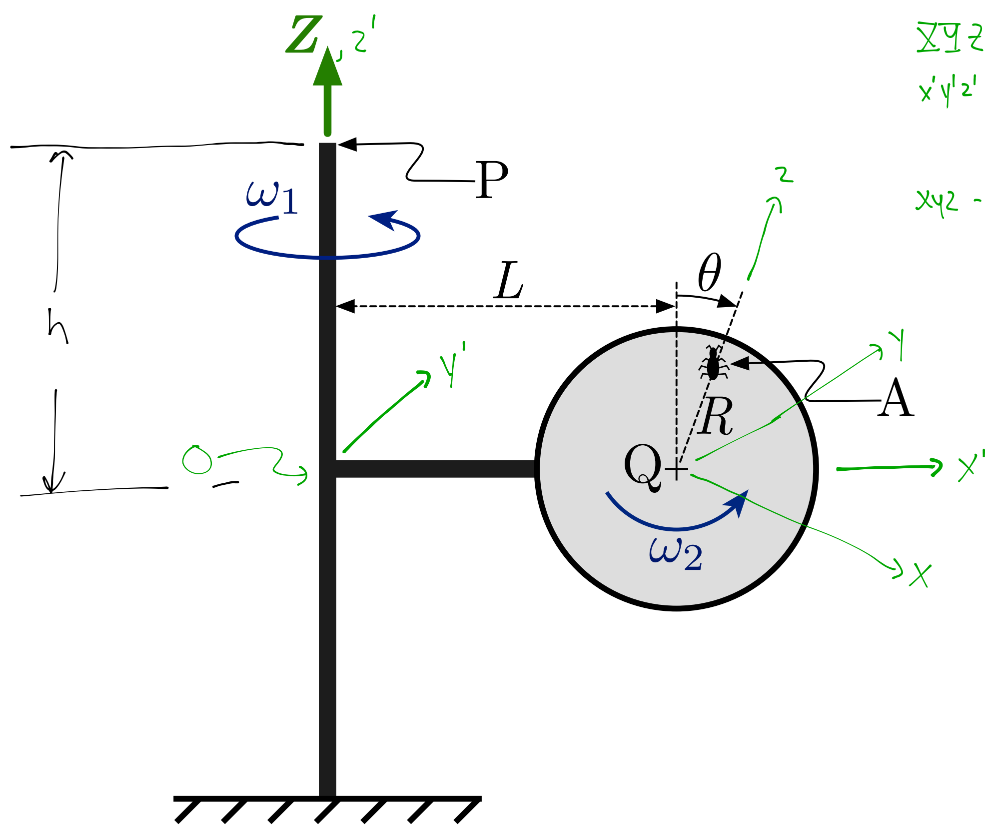
______ **Problem 2 – 30 Points**

Figure 2 shows an instantaneous view of a disk rotating at a rate ω_2 about point Q. An ant is relaxing on the disk (*i.e.* its location on the disk is fixed) at a distance R from its center. When, $\theta = 0$, the ant reaches her topmost position. The center of the disk is located on the T-bar at a distance L from the axis about which the bar is spinning, Z. The rate of rotation of the T-bar is described by ω_1.

a. What is the velocity of the ant, $\bar{v}_A$?

b. What is the acceleration of the ant, $\bar{a}_A$?

c. What does the ant observe as the velocity and acceleration of point P, the top of the T-bar?



XYZ · fixed with Z up

x'y'z' - fixed to T bar with z' up and x' along T

xyz - fixed to disk with z from Q to A

Figure 2: An Ant Riding on a Spinning Disk on a T-bar

a. $\bar{V}_A = \bar{V}_Q + \bar{\omega}_{xyz} \times \bar{r}_{A|Q}$ $\quad \bar{\omega}_{xyz} = \omega_1 \bar{K}' - \omega_2 \bar{J} = \omega_1 \bar{k}' - \omega_2 \bar{j}'$ $\quad \bar{r}_{A|Q} = R\bar{k}$

$\bar{V}_Q = \cancel{\bar{V}_O}^{0} + \bar{\omega}_{x'y'z'} \times \bar{r}_{Q|O}$ $\quad \bar{r}_{Q|O} = L\bar{i}'$ $\quad \bar{\omega}_{x'y'z'} = \omega_1 \bar{k}'$

$\bar{V}_Q = \omega_1 \bar{k}' \times L\bar{i}' = L\omega_1 \bar{j}'$

$$\begin{vmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{vmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{vmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{vmatrix}$$

So

$\bar{V}_A = (L\omega_1 \bar{j}') + (-\omega_2 \bar{j}' + \omega_1 \bar{k}') \times R\bar{k}$

$\bar{V}_A = (L\omega_1 \bar{j}') + (-\omega_2 \bar{j}' + \omega_1 \bar{k}') \times (R\sin\theta\,\bar{i}' + R\cos\theta\,\bar{k}') \leftarrow \bar{k} = s\theta\,\bar{i}' + c\theta\,\bar{k}'$

Problem 2 (cont.)

a. $\bar{V}_A = \left(L\omega_1 \bar{j}'\right) + \left(-\omega_2 \bar{j}' + \omega_1 \bar{k}'\right) \times \left(R\sin\theta\,\bar{i}' + R\cos\theta\,\bar{k}'\right)$

$$\bar{V}_A = \left(L\omega_1 \bar{j}'\right) + \left(R\omega_2 \sin\theta\,\bar{k}' - R\omega_2\cos\theta\,\bar{i}' + R\omega_1 \sin\theta\,\bar{j}'\right)$$

$$\boxed{\bar{V}_A = \left(-R\omega_2\cos\theta\right)\bar{i}' + \left(L\omega_1 + R\omega_1\sin\theta\right)\bar{j}' + \left(R\omega_2\sin\theta\right)\bar{k}'}$$

b. $\bar{a}_A = \bar{a}_Q + \bar{\alpha}_{xyz} \times \bar{r}_{A|Q} + \bar{\omega}_{xyz} \times \left(\bar{\omega}_{xyz} \times \bar{r}_{A|Q}\right)$

$$\bar{a}_Q = \cancel{\bar{a}_O}^{\,0} + \bar{\alpha}_{x'y'z'} \times \bar{r}_{Q/O} + \bar{\omega}_{x'y'z'} \times \left(\bar{\omega}_{x'y'z'} \times \bar{r}_{Q/O}\right) \qquad \bar{\alpha}_{x'y'z'} = \dot{\omega}_1 \bar{k}' + \omega_1 \cancel{\dot{\bar{k}}'}^{\,0}$$

$$a_Q = \left(\dot{\omega}_1 \bar{k}' \times L\bar{i}'\right) + \omega_1 \bar{k}' \times \left(L\omega_1 \bar{j}'\right)$$

$$\bar{a}_Q = L\dot{\omega}_1 \bar{j}' - L\omega_1^2 \bar{i}'$$

$$\bar{\alpha}_{xyz} = \dot{\omega}_1 \bar{k}' + \cancel{\omega_1 \dot{\bar{k}}'}^{\,0} - \dot{\omega}_2 \bar{j}' - \omega_2 \dot{\bar{j}}' = \dot{\omega}_1 \bar{k}' - \dot{\omega}_2 \bar{j}' - \omega_2\left(\bar{\omega}_{x'y'z'} \times \bar{j}'\right)$$

$$= \dot{\omega}_1 \bar{k}' - \dot{\omega}_2 \bar{j}' - \omega_2\left(\omega_1 \bar{k}' \times \bar{j}'\right)$$

$$= \omega_1\omega_2 \bar{i}' - \dot{\omega}_2 \bar{j}' + \dot{\omega}_1 \bar{k}'$$

$$\bar{r}_{A|Q} = R\bar{k} = R\sin\theta\,\bar{i}' + R\cos\theta\,\bar{k}'$$

$$\bar{\omega}_{xyz} \times \left(\bar{\omega}_{xyz} \times \bar{r}_{A|Q}\right) = \bar{\omega}_{xyz} \times \left(R\omega_2\sin\theta\,\bar{k}' - R\omega_2\cos\theta\,\bar{i}' + R\omega_1\sin\theta\,\bar{j}'\right)$$

$$= \left(-\omega_2 \bar{j}' + \omega_1 \bar{k}'\right) \times \left(R\omega_2\sin\theta\,\bar{k}' - R\omega_2\cos\theta\,\bar{i}' + R\omega_1\sin\theta\,\bar{j}'\right)$$

$$\boxed{\text{Collect these terms into } \bar{a}_A = \bar{a}_Q + \bar{\alpha}_{xyz} \times \bar{r}_{A|Q} + \bar{\omega}_{xyz} \times \left(\bar{\omega}_{xyz} \times \bar{r}_{A|Q}\right) \text{ to find } \bar{a}_A}$$

Problem 2 (cont.)

c. Write the velocity and acceleration of point P relative to the cart using the xyz frame

$$\cancel{\bar{v}_P}^{\,0} = \bar{v}_A + (\bar{v}_P)_{xyz} + \bar{\omega}_{xyz} \times \bar{r}_{P/A}$$

$$(\bar{v}_P)_{xyz} = -\bar{v}_A - \bar{\omega}_{xyz} \times \bar{r}_{P/A}$$

from part a of this problem:

$$\bar{v}_A = (-R\omega_2\cos\theta)\,\bar{i}' + (L\omega_1 + R\omega_1\sin\theta)\,\bar{j}' + (R\omega_2\sin\theta)\,\bar{k}'$$

$$\bar{\omega}_{xyz} = \omega_1\bar{k}' - \omega_2\bar{J} = \omega_1\bar{k}' - \omega_2\bar{j}'$$

Define $\bar{r}_{P/A} = -L\bar{i}' + h\bar{k}'$ so $\bar{\omega}_{xyz} \times \bar{r}_{P/A} = (\omega_1\bar{k}' - \omega_2\bar{j}') \times (-L\bar{i}' + h\bar{k}')$

$$= -L\omega_1\bar{j}' + L\omega_2\bar{k}' - h\omega_2\bar{i}'$$

so

$$\boxed{\begin{aligned}(\bar{v}_P)_{xyz} = {} & (R\omega_2\cos\theta + h\omega_2)\,\bar{i}' + \\ & (-\cancel{L\omega_1} - R\omega_1\sin\theta + \cancel{L\omega_1})\,\bar{j}' + \\ & (-R\omega_2\sin\theta + L\omega_2)\,\bar{k}'\end{aligned}}$$

We can follow a similar procedure for acceleration

$$\cancel{\bar{a}_P}^{\,0} = \bar{a}_A + (\bar{a}_P)_{xyz} + \bar{\alpha}_{xyz} \times \bar{r}_{P/A} + \bar{\omega}_{xyz} \times (\omega_{xyz} \times \bar{r}_{P/A}) + 2\omega_{xyz} \times (\bar{v}_P)_{xyz}$$

So

$$\boxed{(\bar{a}_P)_{xyz} = -\left[\bar{a}_A + \bar{\alpha}_{xyz} \times \bar{r}_{P/A} + \bar{\omega}_{xyz} \times (\bar{\omega}_{xyz} \times \bar{r}_{P/A}) + 2\bar{\omega}_{xyz} \times (\bar{v}_P)_{xyz}\right]}$$

where all these terms are defined in earlier parts of this problem.

______ **Problem 3 – 30 Points**

The sketch in Figure 3 shows a thin plate of mass, m, connected to a massless shaft. The corner of the plate is labeled as O, and the shaft-plate body rotates around X at a constant rate, ω. For this system:

a. Write the angular momentum about point O. You do *not* need to derive the inertial properties of the plate, but be sure properly identify what terms would be needed.

b. What moment must be applied at point O for this motion to occur?

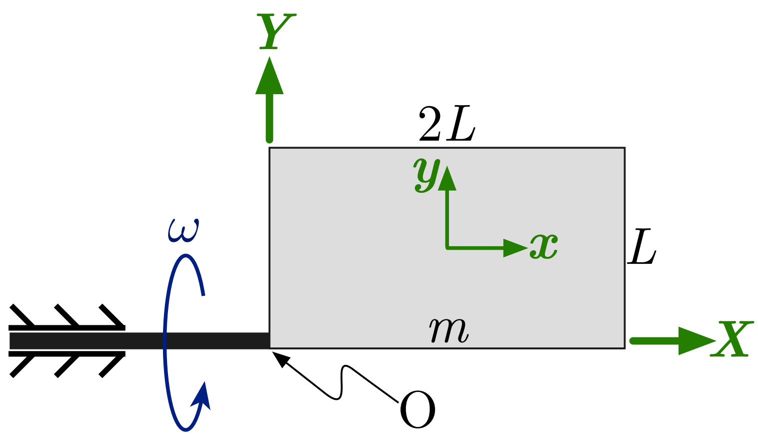


Figure 3: A Rotating Thin Plate

a. A lookup table would likely have the interial properties defined at the centroid of the plate, at the origin of the xyz axes in Figure 3. We need to use the parallel axis theorem to determine the properties at point O, in the XYZ frame.

Let's call the inertial properties at that point

$I_{XX}, I_{YY}, I_{ZZ}, I_{XZ}, I_{XY}, I_{XZ}$

Because this is a thin plate, we can also say that $I_{XZ} = I_{YZ} = 0$

so

$$I = \begin{bmatrix} I_{XX} & -I_{XY} & 0 \\ -I_{XY} & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{bmatrix}$$

The angular velocity of the plate is $\bar{\omega}_{plate} = \omega \bar{I}$

so

$$\boxed{\bar{H}_o = I\bar{\omega} = \begin{bmatrix} I_{XX} & -I_{XY} & 0 \\ -I_{XY} & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = I_{XX}\omega\bar{I} - I_{XY}\omega\bar{J}}$$

Problem 3 (cont.)

b. $\sum \bar{M}_O = \dot{\bar{H}}_O = \cancel{\frac{\partial \bar{H}_O}{\partial t}}^{\,0} + \bar{\omega} \times \bar{H}_O$

$= \omega \bar{I} \times \left[I_{xx} \omega \bar{I} - I_{xy} \omega \bar{J} \right]$

$$\boxed{\sum \bar{M}_O = -I_{xy} \omega^2 \bar{K}}$$