Problem 4.11

4.11. Derive the linearized equation of motion for the 2 DOF system shown in Figure P4.11. Find the natural frequencies and eigenvectors. $m_1 = 30$ kg, $m_2 = 2$ kg, $k_1 = 15$ N/m, $l = 2$ m.

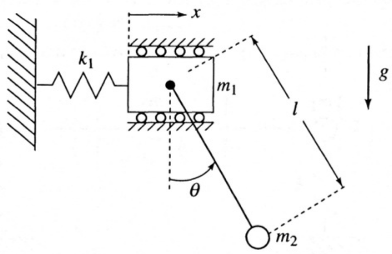


Figure P4.11

You could copy the equations of motion from the notes, but I'll run through it again here for completeness.

I'll use Lagrange's Method.

$$T = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{x} + l\dot{\theta})^2 \leftarrow \text{assuming small angles of } \theta$$

$$V = \tfrac{1}{2} k_1 x^2 - m_2 g l \cos\theta \leftarrow \text{we need to keep the cosine here}$$

$$L = T - V = \left[\tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{x} + l\dot{\theta})^2\right] - \left[\tfrac{1}{2} k x^2 - m_2 g l \cos\theta\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_c}\right) - \frac{\partial L}{\partial q_c} = 0 \leftarrow \begin{array}{l}\text{no damping}\\ \text{no external forces}\end{array}$$

for x

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2(\dot{x} + l\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1 \ddot{x} + m_2(\ddot{x} + l\ddot{\theta})$$

$$\frac{\partial L}{\partial x} = -kx$$

$$(m_1 + m_2)\ddot{x} + m_2 l \ddot{\theta} - kx = 0$$

for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 l(\dot{x} + l\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2 l \ddot{x} + m_2 l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \leftarrow \text{Now assume small angles } \sin\theta \approx \theta$$

$$m_2 l \ddot{x} + m_2 l^2 \ddot{\theta} + m_2 g l \theta = 0$$

In matrix form:

$$\underbrace{\begin{bmatrix} m_1+m_2 & m_2 l \\ m_2 l & m_2 l^2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k & 0 \\ 0 & m_2 g l \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$M\ddot{X} + KX = 0$$

To solve the eigenvalue problem:

$$\det([K - \omega^2 M]) = 0, \text{ then for each } \omega_i \text{ solve } [K - \omega_i^2 M][\bar{X}_i] = 0 \text{ for } \bar{X}_i$$

$$\boxed{\omega_1^2 = 0.47 \left(\tfrac{rad}{s}\right)^2 \quad \omega_2^2 = 5.27 \left(\tfrac{rad}{s}\right)^2 \quad \bar{X}_1 = \begin{bmatrix} 1 \\ 0.05 \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} -0.14 \\ 1 \end{bmatrix}}$$

Problem 4.14

4.14. Determine the eigenvectors and natural frequencies for the system illustrated in Figure P4.14. The bar is uniform. Discuss the physical meaning of the responses. $m_1 = 100$ kg, $m_2 = 1500$ kg, $k_1 = 10,000$ N/m, $k_2 = 12,000$ N/m, $k_3 = 70,000$ N/m, $l = 4$ m.

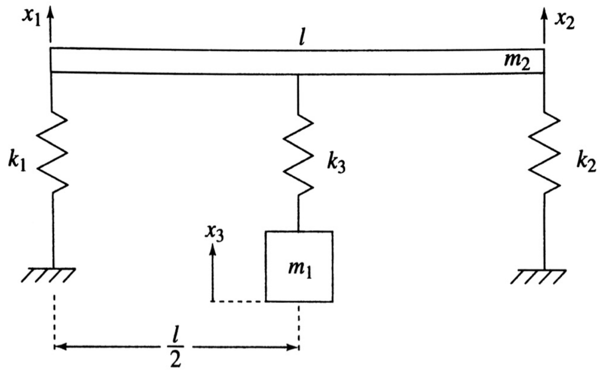
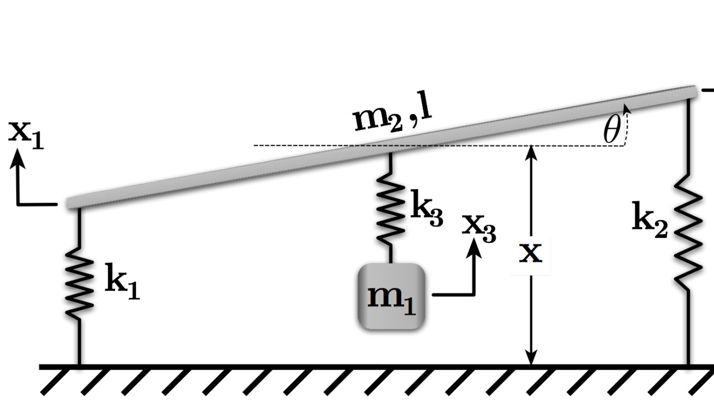


Figure P4.14



Solving via Lagrange's method:

$$T = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I_c \dot{\theta}^2 + \frac{1}{2} m_1 \dot{x}_3^2 \quad \leftarrow I_c \equiv \text{Moment of inertia about the COM} = \frac{1}{2} m l^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_3 x_3^2 \longleftarrow$$ but, we need to write in terms of the generalized coords $\bar{q} = (x, \theta, x_3)$

$$= \frac{1}{2} k_1 \left(x - \frac{l}{2}\sin\theta\right)^2 + \frac{1}{2} k_2 \left(x + \frac{l}{2}\sin\theta\right)^2 + \frac{1}{2} k_3 \left(x - x_3\right)^2$$

Note: We can ignore gravity because all the coordinates are measured from equil.

$L = T - V$

for x

$$\frac{\partial L}{\partial \dot{x}} = m_2 \dot{x} \qquad \frac{\partial L}{\partial x} = -k_1\left(x - \frac{l}{2}\sin\theta\right) - k_2\left(x + \frac{l}{2}\sin\theta\right) - k_3\left(x - x_3\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_2 \ddot{x} \qquad \boxed{m_2 \ddot{x} + (k_1 + k_2 + k_3)x + (k_2 - k_1)\frac{l}{2}\sin\theta - k_3 x_3}$$

for θ

$$\frac{\partial L}{\partial \dot{\theta}} = I_c \dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -k_1\left(x - \frac{l}{2}\sin\theta\right)\left(-\frac{l}{2}\cos\theta\right) - k_2\left(x + \frac{l}{2}\sin\theta\right)\left(\frac{l}{2}\cos\theta\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I_c \ddot{\theta} \qquad \boxed{I_c \ddot{\theta} + (k_2 - k_1)\frac{l}{2} x \cos\theta + (k_2 - k_1)\left(\frac{l}{2}\right)^2 \sin\theta\cos\theta = 0}$$

for x_3

$$\frac{\partial L}{\partial \dot{x}_3} = m_3 \dot{x}_3 \qquad \frac{\partial L}{\partial x_3} = +k_3\left(x - x_3\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = m_3 \ddot{x}_3 \qquad \boxed{m_3 \ddot{x}_3 - k_3 x + k_3 x_3 = 0}$$

Problem 4.14 (cont.)

$$m_2\ddot{x} + (k_1+k_2+k_3)x + (k_2-k_1)\frac{l}{2}\sin\theta - k_3x_3 = 0$$

$$I_c\ddot{\theta} + (k_2-k_1)\frac{l}{2}x\cos\theta + (k_2-k_1)\left(\frac{l}{2}\right)^2\sin\theta\cos\theta = 0$$

$$m_3\ddot{x}_3 - k_3x + k_3x_3 = 0$$

Linearize via small angles

$$m_2\ddot{x} + (k_1+k_2+k_3)x + (k_2-k_1)\frac{l}{2}\theta - k_3x_3 = 0$$

$$I_c\ddot{\theta} + (k_2-k_1)\frac{l}{2}x + (k_2-k_1)\left(\frac{l}{2}\right)^2\theta = 0$$

$$m_3\ddot{x}_3 - k_3x + k_3x_3 = 0$$

In matrix form:

$$\underbrace{\begin{bmatrix} m_2 & 0 & 0 \\ 0 & I_c & 0 \\ 0 & 0 & m_1 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \\ \ddot{x}_3 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k_1+k_2+k_3 & (k_2-k_1)\frac{l}{2} & -k_3 \\ (k_2-k_1)\frac{l}{2} & (k_2-k_1)\left(\frac{l}{2}\right)^2 & 0 \\ -k_3 & 0 & k_3 \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} x \\ \theta \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{0}$$

Now, solve:

1) $\det(K-\omega_i^2M)=0$ for eigenvalues ω_i^2

2) For each ω_i^2, solve $[K-\omega_i^2M]X_i=0$ for eigenvector X_i