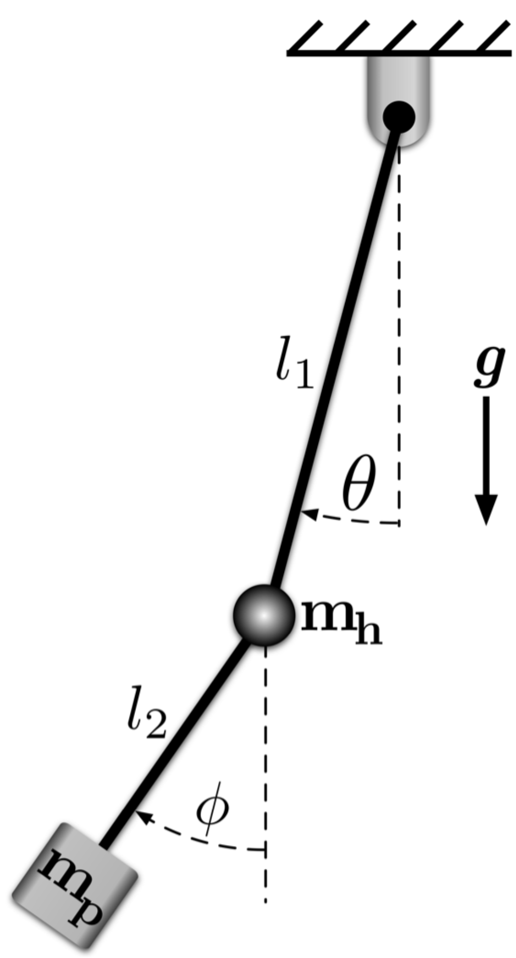
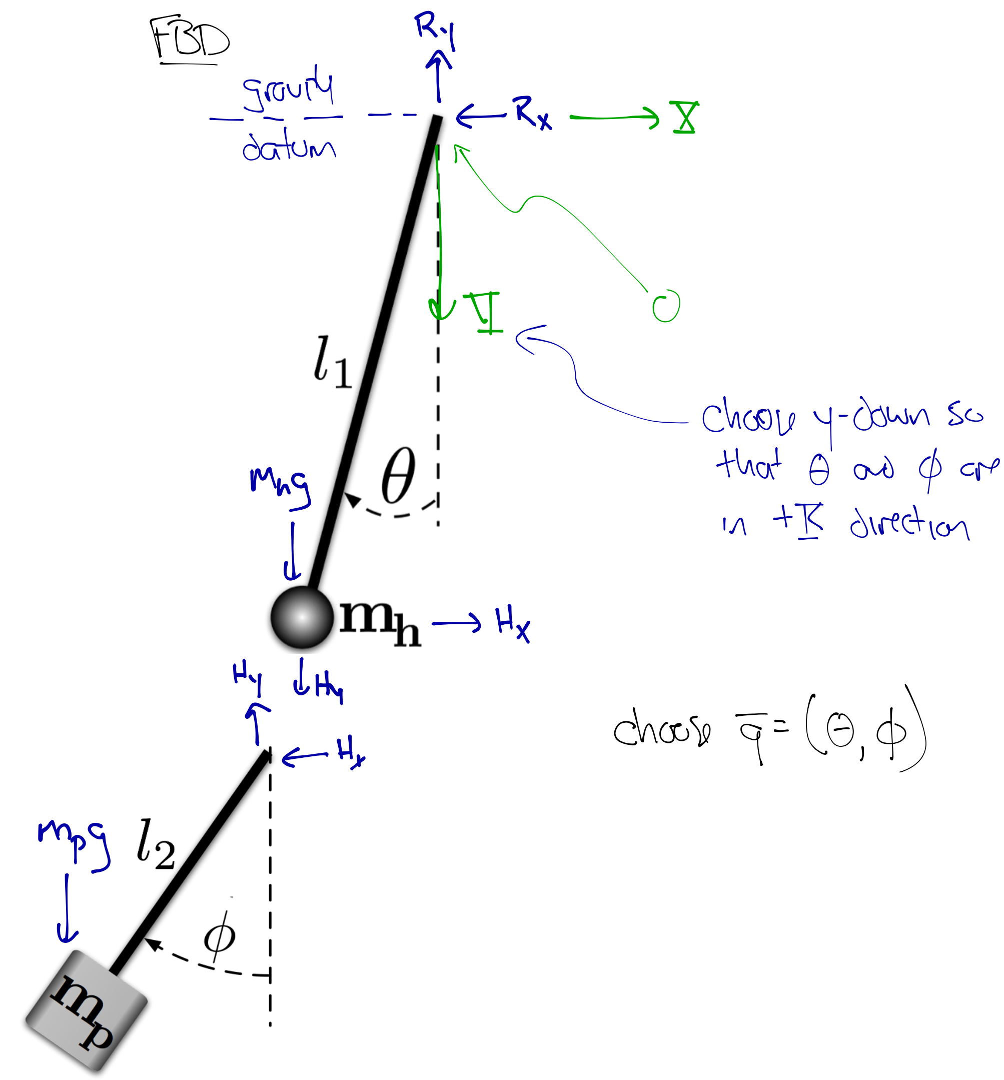
Double Pendulum Modeling





choose $\bar q = (\theta, \phi)$

To avoid needing to account for the reaction forces, let's use Lagrange's Method.

We need to write the total velocities of m_h and m_p to form the kinetic energy equation.

$$T = \frac{1}{2} m_h \bar v_h^T \bar v_h + \frac{1}{2} m_p \bar v_p^T \bar v_p$$

To find those, we can write their absolute positions relative to point O.

$$\bar r_{h/o} = -l_1 \sin\theta\, \bar I + l_1 \cos\theta\, \bar J$$

$$\bar v_h = \frac{d\bar r_{h/o}}{dt} = -l_1 \dot\theta \cos\theta\, \bar I - l_1 \dot\theta \sin\theta\, \bar J$$

$$\bar v_h^T \bar v_h = \bar v_h \cdot \bar v_h = l_1^2 \dot\theta^2 \cos^2\theta + l_1^2 \dot\theta^2 \sin^2\theta$$
$$= l_1^2 \dot\theta^2 \left(\overbrace{\cos^2\theta + \sin^2\theta}^{1}\right)$$

$$\bar r_{p/o} = \bar r_{h/o} + \left(-l_2 \sin\phi\, \bar I + l_2 \cos\phi\, \bar J\right)$$
$$= \left(-l_1 \sin\theta - l_2 \sin\phi\right) \bar I + \left(l_1 \cos\theta + l_2 \cos\phi\right) \bar J$$

$$\bar v_p = \frac{d\bar r_{p/o}}{dt} = \left[-l_1 \dot\theta \cos\theta - l_2 \dot\phi \cos\phi\right] \bar I$$
$$+ \left[-l_1 \dot\theta \sin\theta - l_2 \dot\phi \sin\phi\right] \bar J$$

$$\bar v_p^T \bar v_p = \bar v_p \cdot \bar v_p = \left[+l_1^2 \dot\theta^2 \cos^2\theta + 2 l_1 l_2 \dot\theta \dot\phi \cos\theta \cos\phi + l_2^2 \dot\phi^2 \cos^2\phi\right]$$
$$+ \left[l_1^2 \dot\theta^2 \sin^2\theta + 2 l_1 l_2 \dot\theta \dot\phi \sin\theta \sin\phi + l_2^2 \dot\phi^2 \sin^2\phi\right]$$

Double Pendulum Modeling (cont.)

$$\bar{V}_h^T \bar{V}_h = \bar{V}_h \cdot \bar{V}_h = l_1^2 \dot{\theta}^2$$

$$\bar{V}_p^T \bar{V}_p = \bar{V}_p \cdot \bar{V}_p = \left[+ l_1^2 \dot{\theta}^2 \cos^2\theta + 2 l_1 l_2 \dot{\theta}\dot{\phi} \cos\theta \cos\phi + l_2 \dot{\phi}^2 \cos^2\phi \right] + \left[l_1^2 \dot{\theta}^2 \sin^2\theta + 2 l_1 l_2 \dot{\theta}\dot{\phi} \sin\theta \sin\phi + l_2^2 \dot{\phi}^2 \sin^2\phi \right]$$

$$= l_1^2 \dot{\theta}^2 \left(\underbrace{\cos^2\theta + \sin^2\theta}_{1} \right) + l_2 \dot{\phi}^2 \left(\underbrace{\cos^2\phi + \sin^2\phi}_{1} \right) + 2 l_1 l_2 \dot{\theta}\dot{\phi} \left(\underbrace{\cos\theta\cos\phi + \sin\theta\sin\phi}_{= \cos(\theta - \phi)} \right)$$

$$= l_1^2 \dot{\theta}^2 + l_2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta}\dot{\phi} \cos(\theta - \phi)$$

So,

$$\boxed{T = \frac{1}{2} m_h l_1^2 \dot{\theta}^2 + \frac{1}{2} m_p \left(l_1^2 \dot{\theta}^2 + l_2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta}\dot{\phi} \cos(\theta - \phi) \right)}$$

Now, let's determine the potential energy term of the Lagrangian. We just have gravitational potential for the two masses

$$V = m_h g \left(-l_1 \cos\theta \right) + m_p g \left(-l_1 \cos\theta - l_2 \cos\phi \right)$$

$$\boxed{V = -(m_h + m_p) g l_1 \cos\theta - m_p g l_2 \cos\phi}$$

So

$$\boxed{L = \frac{1}{2} m_h l_1^2 \dot{\theta}^2 + \frac{1}{2} m_p \left(l_1^2 \dot{\theta}^2 + l_2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta}\dot{\phi} \cos(\theta - \phi) \right) + (m_h + m_p) g l_1 \cos\theta + m_p g l_2 \cos\phi}$$

Now, just do the calculus to determine the eq. of motion for each generalized coord.

For $q_1 = \theta \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$

For $q_2 = \phi \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0$

Double Pendulum Modeling (cont.)

$$L = \frac{1}{2}m_h l_1^2 \dot{\theta}^2 + \frac{1}{2}m_p\left(l_1^2\dot{\theta}^2 + l_2\dot{\phi}^2 + 2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right) + (m_h+m_p)gl_1\cos\theta + m_pgl_2\cos\phi$$

For $q_1 = \theta \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{\partial L}{\partial \dot{\theta}} = m_h l_1^2\dot{\theta} + m_p l_1^2\dot{\theta} + m_p l_1 l_2 \dot{\phi}\cos(\theta-\phi) = (m_h+m_p)l_1^2\dot{\theta} + m_pl_1l_2\dot{\phi}\cos(\theta-\phi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (m_h+m_p)l_1^2\ddot{\theta} + m_pl_1l_2\ddot{\phi}\cos(\theta-\phi) - m_pl_1l_2\dot{\phi}(\underline{\dot{\theta}}-\dot{\phi})\sin(\theta-\phi)$$

$$\frac{\partial L}{\partial \theta} = \underline{-m_pl_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi)} - (m_h+m_p)gl_1\sin\theta$$

These cancel

$$\boxed{(m_h+m_p)l_1^2\ddot{\theta} + m_pl_1l_2\ddot{\phi}\cos(\theta-\phi) + m_pl_1l_2\dot{\phi}^2\sin(\theta-\phi) + (m_h+m_p)gl_1\sin\theta = 0}$$

For $q_2 = \phi \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$

$$\frac{\partial L}{\partial \dot{\phi}} = m_pl_2^2\dot{\phi} + m_pl_1l_2\dot{\theta}\cos(\theta-\phi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = m_pl_2^2\ddot{\phi} + m_pl_1l_2\ddot{\theta}\cos(\theta-\phi) - m_pl_1l_2\dot{\theta}(\dot{\theta}-\underline{\dot{\phi}})\sin(\theta-\phi)$$

$$\frac{\partial L}{\partial \phi} = \underline{+m_pl_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi)} - m_pgl_2\sin\phi$$

cancel

$$\boxed{m_pl_2^2\ddot{\phi} + m_pl_1l_2\ddot{\theta}\cos(\theta-\phi) - m_pl_1l_2\dot{\theta}^2\sin(\theta-\phi) + m_pgl_2\sin\phi = 0}$$

Double Pendulum Modeling (cont.)

$$(m_n + m_p)\, l_1^2 \ddot{\theta} + m_p l_1 l_2 \ddot{\phi} \cos(\theta - \phi) + m_p l_1 l_2 \dot{\phi}^2 \sin(\theta - \phi) + (m_n + m_p) g l_1 \sin\theta = 0$$

$$m_p l_2^2 \ddot{\phi} + m_p l_1 l_2 \ddot{\theta} \cos(\theta - \phi) - m_p l_1 l_2 \dot{\theta}^2 \sin(\theta - \phi) + m_p g l_2 \sin\phi = 0$$

Q: How do we get natural freq (and mode shapes) from these equations?

They're nonlinear → need to linearize first

We'll take a somewhat crude approach by just using small angles approx. around $\theta_0 = \phi_0 = 0$

θ is small, ϕ is small, so → $\cos(\theta - \phi) = 1$ and $\sin(\theta - \phi) \approx \theta - \phi$

$\sin\theta \approx \theta$ and $\sin\phi \approx 1$

Q: What other problematic terms are there?

$\dot{\phi}^2$ and $\dot{\theta}^2$ → we'll assume that if θ and ϕ are small, then

$\dot{\theta}$ and $\dot{\phi}$ are also small → $\dot{\theta}^2 \approx 0$ and $\dot{\phi}^2 \approx 0$

So, our linearized equations of motion are:

$$\boxed{\begin{aligned} (m_n + m_p)\, l_1^2 \ddot{\theta} + m_p l_1 l_2 \ddot{\phi} + (m_n + m_p) g l_1 \theta = 0 \\ m_p l_2^2 \ddot{\phi} + m_p l_1 l_2 \ddot{\theta} + m_p g l_2 \phi = 0 \end{aligned}}$$

Or, written in matrix form

$$\boxed{\begin{bmatrix} (m_n + m_p) l_1^2 & m_p l_1 l_2 \\ m_p l_1 l_2 & m_p l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} (m_n + m_p) g l_1 & 0 \\ 0 & m_p g l_2 \end{bmatrix} \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

Double Pendulum Modeling (cont.)

$$\underbrace{\begin{bmatrix} (m_h+m_p)l_1^2 & m_p l_1 l_2 \\ m_p l_1 l_2 & m_p l_2^2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} (m_h+m_p) g l_1 & 0 \\ 0 & m_p g l_2 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} \theta \\ \phi \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

To find the (linearized) natural freq and mode-shapes, solve the eigenvalue problem for this system

for eigenvalues $\rightarrow \omega_i^2 \rightarrow$ solve $\det\left([K-\omega_i^2 M]\right)=0$ for ω_i^2

for eigenvectors $\rightarrow$ for each eigenvalue ω_i^2 solve $[K-\omega_i^2 M]X_i=0$ for X_i