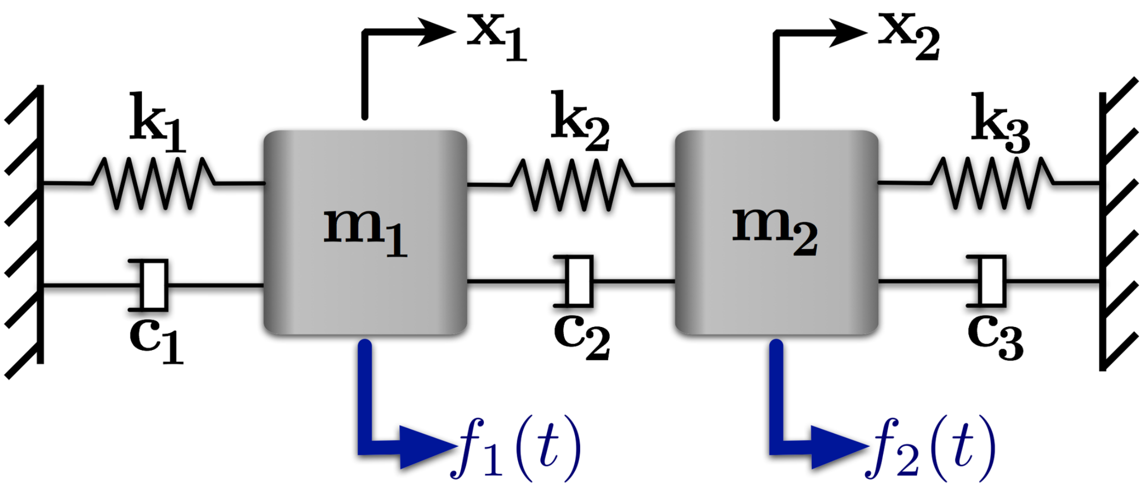
Linear Damping (Sec 4.10)



Q: What do you expect the equations of motion to look like?

$$M\ddot{X} + C\dot{X} + KX = F$$

What does C look like?
(it should mirror K, since each spring/damper pair is connected in parallel)

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{F}$$

For the undamped version of this system we found the mode shapes to be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ when $m_1 = m_2$ and $k_1 = k_2 = k_3$.

Q: If we have initial conditions $x_1(0) = 1$, $x_2(0) = 1$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$, do we expect to still only get the 1st mode?

Not necessarily... it depends on c_1, c_2, and c_3.

Q: In general, when do the modes match the undamped case?

$M\ddot{X} + C\dot{X} + KX = 0$ — Assume $X(t) = \bar{X} e^{\lambda t}$ ← remember that $\lambda = \omega_i^2$ for the undamped case

λ is complex

$[\lambda^2 M + \lambda C + K]\bar{X} = 0$ ← More difficult to solve than the undamped case

Q: How can we solve this in Python (and most other scientific computing packages)?

Almost all expect the problem in form of:

$\dot{X} = AX$ or $A\dot{X} = BX$ ← So, we need to write of equation of motion as a series of 1st order ODEs.

We can use:

1) State-Space form - $\dot{x} = Ax + Bu$
2) Symmetry to rewrite as $B\dot{Y} = AY$ where

$$B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix}, \text{ and } Y = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

State-Space Form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = f_1 \longrightarrow \ddot{x}_1 = \frac{1}{m_1}\left[-(k_1+k_2)x_1 + k_2x_2 - (c_1+c_2)\dot{x}_1 + c_2\dot{x}_2 + f_1\right]$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2+c_3)\dot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = f_2 \longrightarrow \ddot{x}_2 = \frac{1}{m_2}\left[k_1x_1 - (k_2+k_3)x_2 + c_2\dot{x}_1 - (c_2+c_3)\dot{x}_2 + f_2\right]$$

Let $\bar{y} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$, then $\dot{\bar{y}} = \left.\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix}\right\}$ write this in matrix form as $\dot{\bar{y}} = A\bar{y}$

$$\dot{\bar{y}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & \frac{-(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{-(c_2+c_3)}{m_2} \end{bmatrix}\bar{y} + \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

To solve the eigenvalue/eigenvector problem, we ignore any inputs (just like we did when solving for natural frequency for a 1 DoF system). So, we will solve:

$\dot{\bar{y}} = A\bar{y}$ to find eigenvalues and eigenvectors (command is usually something like eig(A))

Symmetric Formulation

Define $B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}$, $A = \begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix}$, and $\bar{y} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$

$$B = \begin{bmatrix} -(k_1+k_2) & k_2 & 0 & 0 \\ k_2 & -(k_2+k_3) & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}$$

$-K$ $\quad$ M

$$A = \begin{bmatrix} 0 & 0 & -(k_1+k_2) & k_2 \\ 0 & 0 & k_2 & -(k_2+k_3) \\ -(k_1+k_2) & k_2 & -(c_1+c_2) & c_2 \\ k_2 & -(k_2+k_3) & c_2 & -(c_2+c_3) \end{bmatrix}$$

$-K$ $\quad$ $-K$ $\quad$ $-C$

For this formulation the command is usually something like eig(A,B)

(A replacing K "position" and B replacing M "position" from the undamped case)

A few points about State-Space or Symmetric Form Solutions

- The eigenvalues returned are ω_i , not ω_i^2
- Have to pick the right elements of the resulting eigenvectors
  (They now include velocity terms)

$$\underbrace{\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\text{example symmetric form}} \quad \text{or} \quad \underbrace{\bar{X} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}}_{\text{example state space form}}$$

pick the terms corresponding to generalized coords

Comparison of Damped Eigensolutions (Sec 4.11)

Q: Do you expect damped eigenvalues and eigenvectors to match undamped?

Only in some special cases.

Proportional Damping

$C = \alpha M + \beta K$, where α and β are real, scalar constants

To understand, let's look at the normal form of $M\ddot{X} + C\dot{X} + KX = 0$

Use $X = UH$ and pre-multiply by U^T (just as we did in the undamped case)

$$\underbrace{U^T M U}_{I}\ddot{H} + \underbrace{U^T C U}_{?}\dot{H} + \underbrace{U^T K U}_{\Omega^2}X = 0$$

$$\ddot{H} + U^T[\alpha M + \beta K]U\dot{H} + \Omega^2 H = 0$$ ← now, distribute U^T and U

$$\ddot{H} + [\alpha \underbrace{U^T M U}_{I} + \beta \underbrace{U^T K U}_{\Omega^2}]\dot{H} + \Omega^2 H = 0$$

← These matrices are diagonal, so the system is still decoupled

Q: Do you expect to (always) see proportional damping in the "real" world?

No. So, what happens then?

We don't get eigenvectors that can decouple the system.
(Ex 4.15 in the book)

"Forced" Decoupling

Just ignore the off-diagonal terms. (Ex 4.16 in the book)

In general, we can do this when the off diagonal terms are small compared to those on the diagonal.

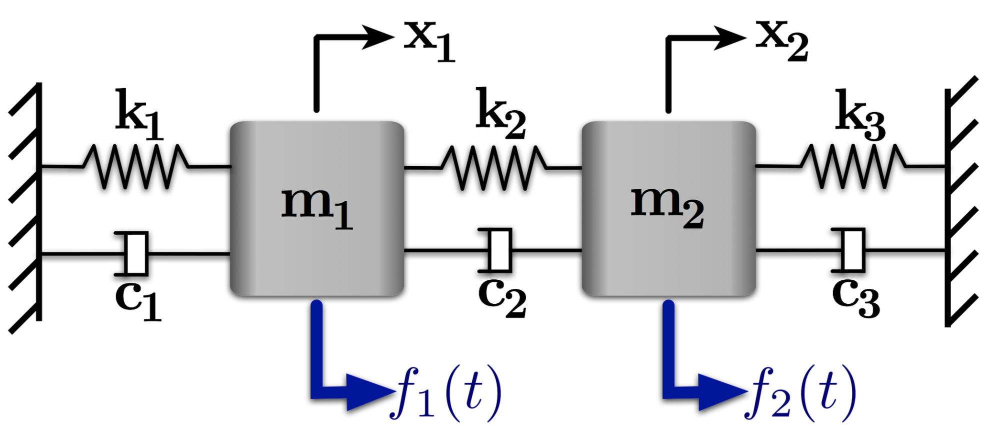
What is small?... it depends on the precision you need.

Forced Response of Damped System (Sec 4.12)

$M\ddot{X} + C\dot{X} + KX = F$ let $F = \bar{F}e^{i\omega t}$, so assume $X = \bar{X}e^{i\omega t}$

$$[-\omega^2 M + i\omega C + K]\bar{X} = \bar{F} \longrightarrow \bar{X} = [-\omega^2 M + i\omega C + K]^{-1}\bar{F}$$

Example 4.18



$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{F}$$

$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos(3t)$ ← note that this is just the real part of e^{3it}

$\omega = 3$

$$X = [-\omega^2 M + i\omega C + K]^{-1}\bar{F}$$

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} -\omega^2 m_1 + i\omega(c_1+c_2) + (k_1+k_2) & i\omega c_2 - k_2 \\ i\omega c_2 - k_2 & -\omega^2 m_2 + i\omega(c_2+c_3) + (k_2+k_3) \end{bmatrix}^{-1} \overbrace{\begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \end{bmatrix}}^{=\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \underbrace{\begin{bmatrix} 0.0604 - 0.01431i \\ -0.1523 + 0.0219i \end{bmatrix}}_{\text{But the response isn't imaginary!!!}}$$

Take the real part (to match the "realness" of the input)

we know $x(t) = \bar{X}_r \cos\omega t - \bar{X}_c \sin\omega t$ ← See p95 for a refresher

$\bar{X} = \bar{X}_r + i\bar{X}_c$ so

$x(t) = (\bar{X}_r + i\bar{X}_c)\overbrace{(\cos\omega t + i\sin\omega t)}^{e^{i\omega t}}$

$= \bar{X}_r\cos\omega t + i\bar{X}_r\sin\omega t + i\bar{X}_c\cos\omega t - \bar{X}_c\sin\omega t$

take the real part

So,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.0604 \\ -0.1523 \end{bmatrix}\cos 3t - \begin{bmatrix} -0.01431 \\ 0.0219 \end{bmatrix}\sin 3t$$