State-space Form

The form of the equations of motion we're getting are not very "simulation friendly"

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}$$

The vibration absorber system

Let's write them in state-space form instead

a system of 1^{st} order ODEs ← Used a lot in controls

We want to write $\ddot{x}$ and $\ddot{\theta}$ in terms of only $x_1, \dot{x}_1, x_2$, and $\dot{x}_2$ — The states of the system

The state vector is usually written as $\bar{x}$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The subscripts here denote the state number... not x_1 and x_2 from eq. of motion.

Can be confusing, so let's use $\bar{w}$ for states

$$\bar{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \longrightarrow \text{We want to write } \dot{\bar{w}} = A\bar{w} + B\bar{u}$$

x_1 and x_2 from eq of motion

Let $$\bar{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

So $$\dot{\bar{w}} = \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \\ \dot{w}_4 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ (\text{just } m_1 \text{ eq of motion}) \\ w_4 \\ (m_2 \text{ eq of motion}) \end{bmatrix} = \begin{bmatrix} w_1 \\ \frac{-(k_1+k_2)}{m_1}w_1 + \frac{k_2}{m_1}w_3 + \frac{f}{m_1} \\ w_3 \\ \frac{k_2}{m_2}w_1 - \frac{k_2}{m_2}w_3 \end{bmatrix}$$

← Now write in matrix form

$$\dot{\bar{w}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}}_{A} \bar{w} + \underbrace{\begin{bmatrix} 0 \\ 1/m \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{f}_{\bar{u}}$$

A = state transition matrix

State-space Form Example (cont.)

Let's look at a different choice for $\bar{w}$

Let

$$\bar{w} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

So

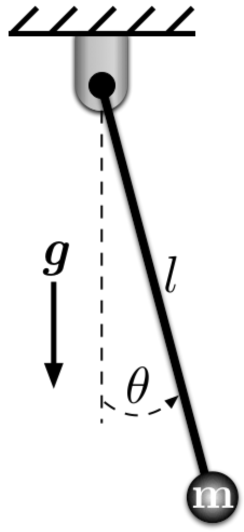
$$\dot{\bar{w}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} w_3 \\ w_4 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} w_3 \\ w_4 \\ -\frac{(k_1+k_2)}{m_1} w_1 + \frac{k_2}{m_1} w_2 + \frac{f}{m_1} \\ \frac{k_2}{m_2} w_1 - \frac{k_2}{m_2} w_2 \end{bmatrix}$$

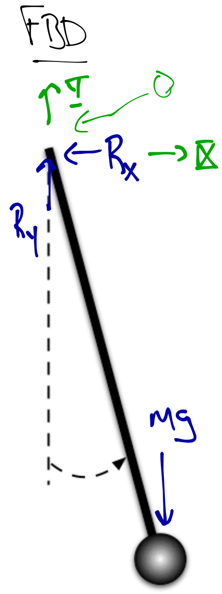
Now, write in matrix form:

$$\dot{\bar{w}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix} \bar{w} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} f$$

State-space Form Example

Let's write the eq of motion for a simple pend. in state-space form.





$\sum \bar{M}_0 = I_0 \bar{\alpha}$

$\sum \bar{M}_0 = \bar{r}_{m/0} \times -mg\bar{J} = (l\sin\theta \bar{I} - l\cos\theta \bar{J}) \times (-mg\bar{J}) = -mgl\sin\theta \bar{K}$

$I_0\bar{\alpha} = ml^2\ddot{\theta}\bar{K}$

So, $\left[ml^2\ddot{\theta} = -mgl\sin\theta\right]\bar{K}$ ← all in the $\bar{K}$ direction, so we'll leave that off from here on

$ml^2\ddot{\theta} + mgl\sin\theta = 0$ ← To write in state-space form, we need to write this as a system of linear 1st order diff eqs.

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

$\left[\ddot{\theta} = -\frac{g}{l}\sin\theta\right]$ ← We can also rewrite as a system of 1st-order nonlinear diff eq and often will do so for nonlinear simulation

Linearized (Matrix) Form

Linearized eq. → $\ddot{\theta} + \frac{g}{l}\theta = 0 \quad \left(\ddot{\theta} = -\frac{g}{l}\theta\right)$

Define $\bar{w} = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$, write $\dot{\bar{w}} = A\bar{w} + B\bar{u}$ → $\dot{\bar{w}} = \begin{bmatrix} w_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} w_2 \\ -\frac{g}{l}w_1\end{bmatrix}$

So $\dot{\bar{w}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}\bar{w}$ ← $B=0$ because there are no nonconservative forcing forcing func.

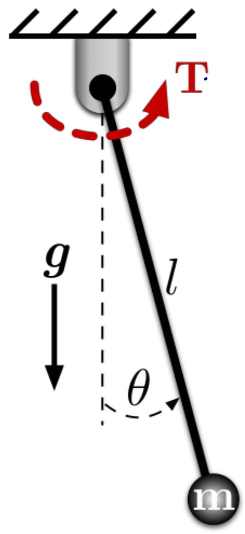
Nonlinear Diff Eq

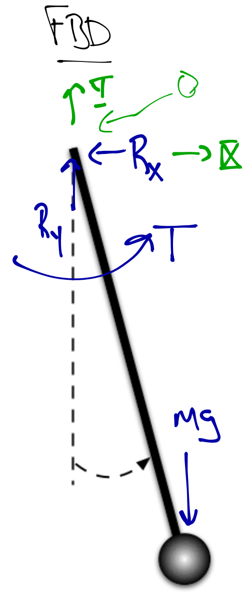
- use the full, nonlinear form of the eq. of motion

Define $\bar{w} = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$, write $\dot{\bar{w}} = f(w,t) = \begin{bmatrix}\dot{w}_1 \\ \dot{w}_2\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} w_2 \\ -\frac{g}{l}\sin(w_1)\end{bmatrix}$

State-space Form Example

Let's write the eq of motion for a simple pend. in state-space form.





$\sum \bar{M}_O = I_O \bar{\alpha}$

$\sum \bar{M}_O = \bar{r}_{m/O} \times -mg\bar{J} + T\bar{K} = (l\sin\theta \bar{I} - l\cos\theta \bar{J}) \times (-mg\bar{J}) + T\bar{K} = [-mgl\sin\theta + T]\bar{K}$

$I_O \bar{\alpha} = ml^2 \ddot{\theta} \bar{K}$

So, $[ml^2\ddot{\theta} = -mgl\sin\theta + T]\bar{K}$ ← all in the $\bar{K}$ direction, so we'll leave that off from here on

$ml^2\ddot{\theta} + mgl\sin\theta = T$ ← To write in state-space form, we need to write this as a system of linear 1st order diff eqs.

$\ddot{\theta} + \frac{g}{l}\sin\theta = T/ml^2$

$\left[\ddot{\theta} = -\frac{g}{l}\sin\theta + T/ml^2\right]$

We can also rewrite as a system of 1st-order nonlinear diff eq and often will do so for nonlinear simulation

Linearized (Matrix) Form

Linearized eq. → $\ddot{\theta} + \frac{g}{l}\theta = 0$ $\left(\ddot{\theta} = -\frac{g}{l}\theta + T/ml^2\right)$

Define $\bar{w} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, write $\dot{\bar{w}} = A\bar{w} + B\bar{u}$ → $\dot{\bar{w}} = \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} w_2 \\ -\frac{g}{l}w_1 + T/ml^2 \end{bmatrix}$

So $\dot{\bar{w}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \bar{w} + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} T$

Nonlinear Diff Eq

- use the full, nonlinear form of the eq. of motion

Define $\bar{w} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, write $\dot{\bar{w}} = f(w,t) = \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} w_2 \\ -\frac{g}{l}\sin(w_1) + T/ml^2 \end{bmatrix}$

State-space Form (Slightly More Complex) Example

The form of the equations of motion we're getting are not very "simulation friendly"

$$\underbrace{\begin{bmatrix} m_1+m_2 & -m_2 l \\ -m_2 l & m_2 l^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & m_2 g l \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}}_{\text{The spring-pendulum system}}$$

Let's write them in state-space form instead ← a system of 1st order ODEs ← Used a lot in controls

We want to write $\ddot{x}$ and $\ddot{\theta}$ in terms of only $\underbrace{x, \dot{x}, \theta, \text{and } \dot{\theta}}_{\text{The states of the system}}$

$$\ddot{x} + \frac{k}{m_1+m_2}x - \frac{m_2}{m_1+m_2} l\ddot{\theta} = \frac{f}{m_1+m_2}$$

$$\ddot{x} = \frac{-k}{m_1+m_2}x + \frac{m_2}{m_1+m_2} l\ddot{\theta} + \frac{f}{m_1+m_2}$$

← we know $\ddot{\theta} = \frac{1}{l}\ddot{x} - \frac{g}{l}\theta$, sub into the $\ddot{x}$ equation

$$\ddot{x} = \frac{-k}{m_1+m_2}x + \frac{m_2}{m_1+m_2} l\left(\frac{1}{l}\ddot{x} - \frac{g}{l}\theta\right) + \frac{f}{m_1+m_2}$$

$$\ddot{x} = \frac{-k}{m_1}x - \frac{m_2}{m_1}g\theta + f$$

Follow a similar procedure for $\ddot{\theta}$ equation to find:

$$\ddot{\theta} = \frac{-k}{m_1 l}x - \left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l}\theta + \frac{f}{l}$$

Now, define a state matrix $\bar{w}$ as

$$\bar{w} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \quad \text{so} \quad \dot{\bar{w}}: \begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = \ddot{x} \\ \dot{w}_3 = w_4 \\ \dot{w}_4 = \ddot{\theta} \end{cases}$$

write this in matrix form

$$\underbrace{\dot{\bar{w}}}_{\dot{X}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & 0 & -\frac{m_2}{m_1}g & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 l & 0 & -\left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l} & 0 \end{bmatrix}}_{A} \underbrace{\bar{w}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1/l \end{bmatrix}}_{B} \underbrace{f(t)}_{U}$$