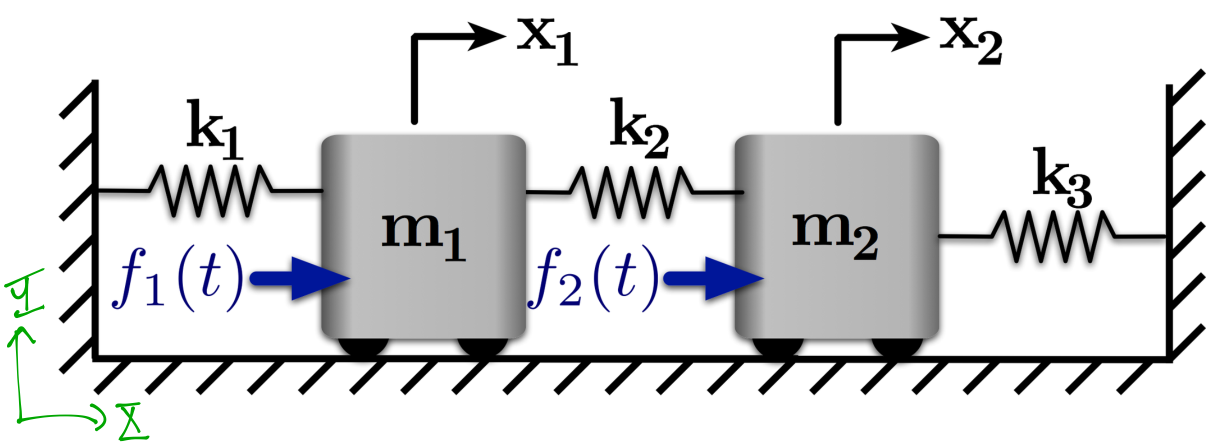
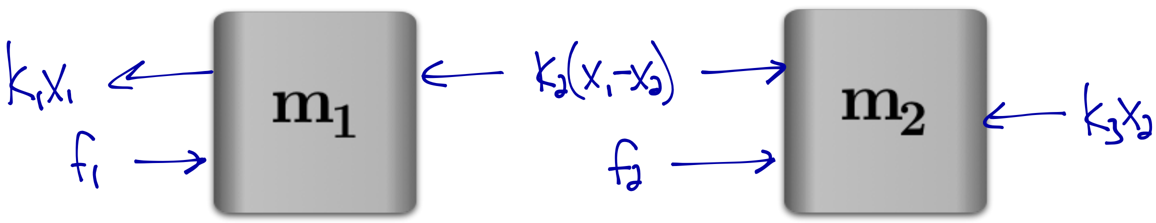
Multi-DOF Undamped Forced Response (Sec. 4.3)



FBD



$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) + f_1$

$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = f_1$

$m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3x_2 + f_2$

$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = f_2$

Write this in matrix form:

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{F}$$

$$M\ddot{X} + KX = F$$

Q: How do we solve this?

1) Assume $f_1 = \bar{f}_1 \cos\omega t$ and $f_2 = \bar{f}_2 \cos\omega t$

2) Assume $x_1 = \bar{x}_1 \cos\omega t$ and $x_2 = \bar{x}_2 \cos\omega t$

Differentiate and sub into the equation of motion.

$$-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \end{bmatrix}$$

$$[K - \omega^2 M]\bar{X} = \bar{F}$$

Now, to have a solution $[K - \omega^2 M]$ must be invertible

Q: See any problems with this?

Matrix inversion is computationally expensive

Q: What's the inverse of a 2x2 matrix?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Aside: We could also use Gaussian Elim or LU decomposition to find the inverse.

Back to our problem:

$$[K-\omega^2 M]\bar{X} = \bar{F} \longrightarrow \bar{X} = [K-\omega^2 M]^{-1}\bar{F}$$

$$= \begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2+k_3-\omega^2 m_2 \end{bmatrix} \bar{F}$$

$$= \frac{1}{\det(K-\omega^2 M)} \begin{bmatrix} k_2+k_3-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix} \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \end{bmatrix}$$

Let $\Delta(\omega) = \det([K-\omega^2 M]) = \underbrace{(k_1+k_2-\omega^2 m_1)(k_2+k_3-\omega^2 m_2)}_{\text{characteristic polynomial}}$ ← Q: Look familiar?

Solving for $\bar{X}_1$ and $\bar{X}_2$, we find

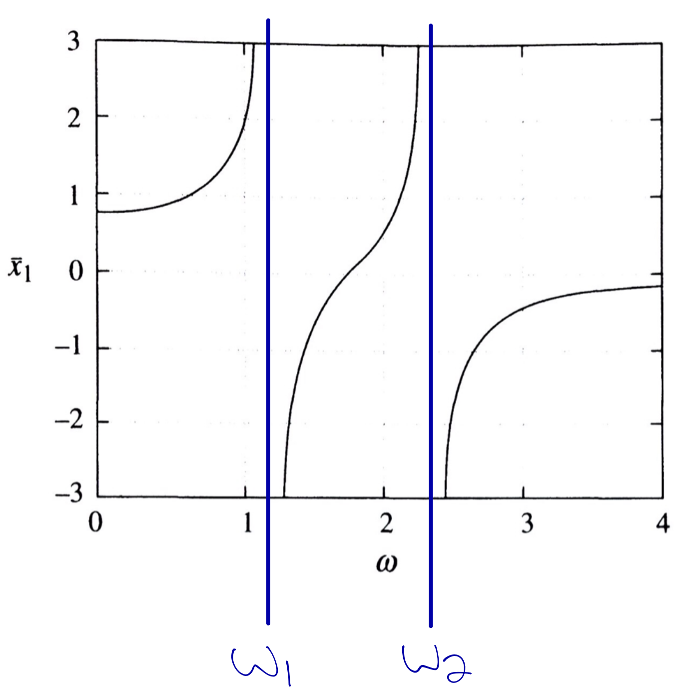
$$\bar{X}_1 = \frac{(k_2+k_3-\omega^2 m_2)\bar{f}_1 + k_2\bar{f}_2}{\Delta(\omega)}$$

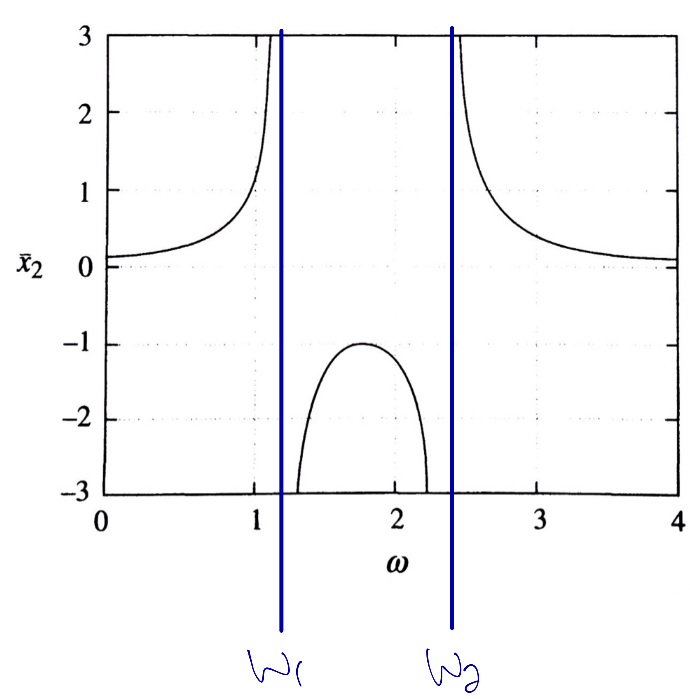
$$\bar{X}_2 = \frac{k_2\bar{f}_1 + (k_1+k_2-\omega^2 m_1)\bar{f}_2}{\Delta(\omega)}$$

Notice that both denominators are the characteristic poly.

Q: If both den. are the char. poly., what can we say about the response?

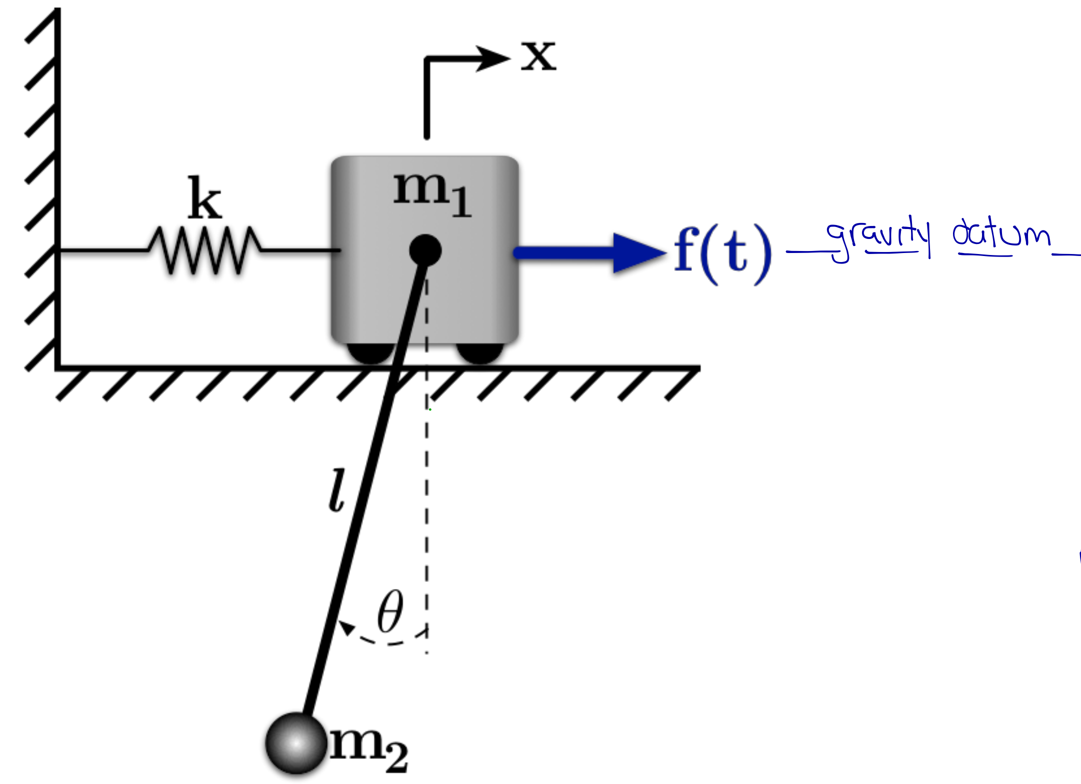
1) den. = 0 at the natural freq.
2) if num ≠ 0 at the natural freq, then the response goes to ∞ at each nat. freq.

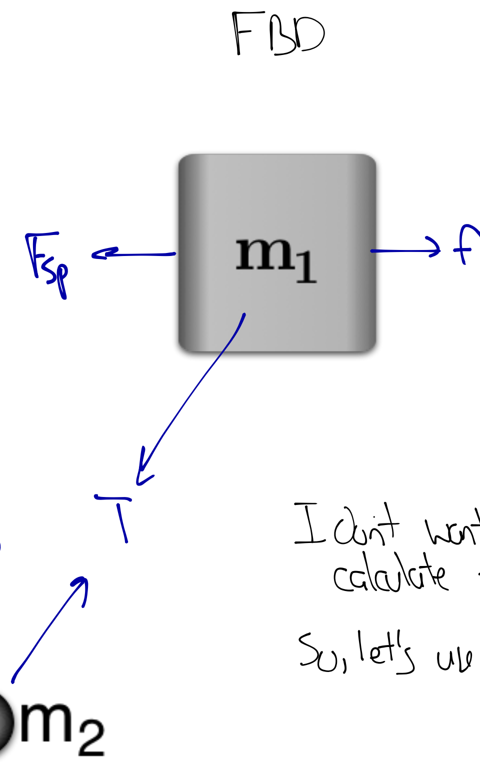




Also notice the phase change at each natural frequency.

Example Multi-DOF Vibration





I don't want (or need) to calculate the tension, T.

So, let's use Lagrange's Method.

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{x} - l\dot{\theta}\right)^2$$ ← assuming small angles of θ about $\theta = 0$

$$V = \frac{1}{2} k x^2 - m_2 g l \cos\theta$$

$$L = T - V = \left[\frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x} - l\dot{\theta})^2\right] - \left[\frac{1}{2} k x^2 - m_2 g l \cos\theta\right]$$

x-equations

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2 (\dot{x} - l\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1 + m_2)\ddot{x} - m_2 l \ddot{\theta}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\boxed{(m_1 + m_2)\ddot{x} - m_2 l \ddot{\theta} + kx = f(t)}$$

f(t) is only acting in δx direction so $Q_1 = f(t)$ and $Q_2 = 0$

θ-equations

$$\frac{\partial L}{\partial \dot{\theta}} = -m_2 l (\dot{x} - l\dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -m_2 l \ddot{x} + m_2 l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -m_2 g l \sin\theta$$ ← assume small angle - $\sin\theta \approx \theta$

$$= -m_2 g l \theta$$

$$\boxed{-m_2 l \ddot{x} + m_2 l^2 \ddot{\theta} + m_2 g l \theta = 0}$$

Now, write in matrix form:

$$\underbrace{\begin{bmatrix} m_1 + m_2 & -m_2 l \\ -m_2 l & m_2 l^2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k & 0 \\ 0 & m_2 g l \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f(t) \\ 0 \end{bmatrix}}_{F}$$

Example Multi-DOF Vibration (cont)

$$\underbrace{\begin{bmatrix} m_1+m_2 & -m_2 l \\ -m_2 l & m_2 l^2 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k & 0 \\ 0 & m_2 g l \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f(t) \\ 0 \end{bmatrix}}_{F}$$

To find natural freq., mode shapes, and response:

- for natural freq: solve $\det\left([K-\omega^2 M]\right)=0$
- for mode shapes, solve $(K-\omega^2 M)\bar{X}=0$ for each ω^2
- for the response, solve $\bar{X}=(K-\omega^2 M)^{-1}\bar{F}$

See Jupyter Notebook for responses of this system.