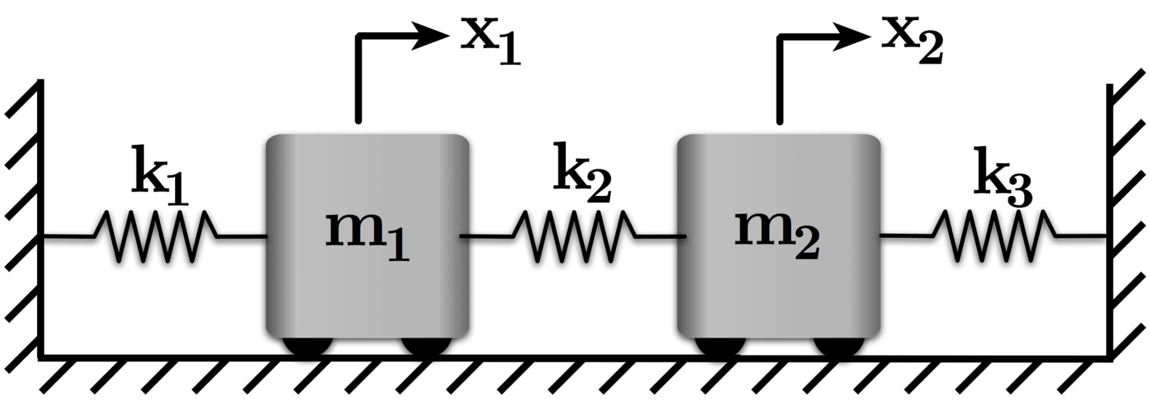
Chapter 4 - Multi-DOF Vibration



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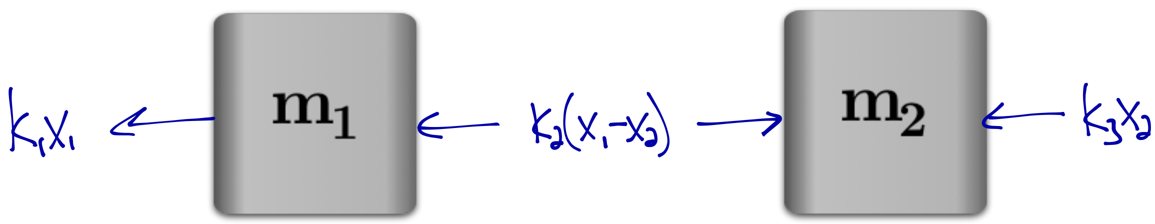

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1-x_2) \qquad m_2\ddot{x}_2 = k_2(x_1-x_2) - k_3x_2$$

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \qquad m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0$$

Write this in matrix form:

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

$$M\ddot{X} + KX = 0 \longleftarrow \text{Look familiar?}$$

Q: How do we solve this?

Just like the scalar case, we can assume a solution:

$$X(t) = \bar{X}e^{i\omega t} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} e^{i\omega t}$$

Differentiate twice and sub into the eq. of motion:

$$M\omega^2\bar{X}\cancel{e^{i\omega t}} + K\bar{X}\cancel{e^{i\omega t}} = 0 \longrightarrow \underbrace{[K - \omega^2M]\bar{X} = 0}$$

This is an eigenvalue problem

form $A(\lambda)X = 0$

Matrix A is a function of λ

$[K-\omega^2 M]\bar{X}=0$

We want to find $\bar{X}$, so we can get $X(t)$.

Q: What's one solution?

$\bar{X}=0$ ← trivial solution = no motion

Another Solution:

let $A=K-\omega^2 M \rightarrow A\bar{X}=0$ ← we need A to not be invertable

if A is invertable, then

$A^{-1}A\bar{X}=A^{-1}0$ ← The trivial solution again! So...

Q: When is A not invertable?

(linear dependent rows or columns – not full rank)

$\det(A)=0 \longrightarrow \det[K-\omega^2 M]=0$

$$\det\left(\begin{bmatrix}(k_1+k_2)-\omega^2 m_1 & -k_2 \\ -k_2 & (k_2+k_3)-\omega^2 m_2\end{bmatrix}\right)=0$$

Q: What's the $\det\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)$?

$ad-bc$

$$\underbrace{\underbrace{\left((k_1+k_2)-\omega^2 m_1\right)\left((k_2+k_3)-\omega^2 m_2\right)-k_2^2}_{\text{characteristic polynomial}}=0}_{\text{characteristic equation}}$$

characteristic equation ← The roots of this define the natural frequencies.

Controls Preview/Review:
Solutions to this are also called the poles of the system. Their location in the s-plane tells us about the system.

Let's look at some example parameters:

$m_1=m_2=1\,kg$

$k_1=k_2=k_3=4\,N/m$

char. eq → $\omega^4-16\omega^2+48=0$ ← solve this for ω^2 (just quadratic in ω^2 = easy)

$$\omega_{1,2}^2=\frac{1}{2}\left(16\pm\sqrt{16^2-4(48)}\right)$$

$\omega_1^2=4\left(\frac{rad}{s}\right)^2$

$\omega_2^2=12\left(\frac{rad}{s}\right)^2$

1. Label freq. from lowest up
2. These are eigenvalues
3. Square root of these is the natural freq.

Now, we need to find $\bar{X}$ (The eigenvectors)

To do so, plug in the eigenvalues and solve for $\bar{X}$

Look at ω_1^2 first:

$$\left(\begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \omega_1^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\omega_1^2 = 4$ so,

Both equations have the same solution b/c rows are linearly dep when det = 0

$$\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} 4\bar{x}_1 - 4\bar{x}_2 = 0 \\ -4\bar{x}_1 + 4\bar{x}_2 = 0 \end{matrix} \Bigg\} \; \bar{x}_1 = \bar{x}_2 \rightarrow \bar{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, the part of the solution from this eigenvalue/eigenvector pair is:

$$X(t) = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_1 t} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\omega_1 t} \quad \text{or} \quad X(t) = b_1 \bar{X}_1 \cos(\omega_1 t - \phi_1)$$

Now look at ω_2^2:

$$\left(\begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \omega_2^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{vmatrix} \bar{x}_1 \\ \bar{x}_2 \end{vmatrix} = 0$$

$$\omega_2^2 = 12 \left(\frac{\text{rad}}{s}\right)^2$$

$$\begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = 0 \quad \begin{matrix} -4\bar{x}_1 \cdot 4\bar{x}_2 = 0 \\ -4\bar{x}_1 \cdot 4\bar{x}_2 = 0 \end{matrix} \Bigg\} \; \bar{x}_2 = -\bar{x}_1 \longrightarrow \bar{X}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So, the solution from this eigenvalue/eigenvector pair is:

$$X(t) = a_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega_2 t} + a_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\omega_2 t} \quad \text{or} \quad X(t) = b_2 \bar{X}_2 \cos(\omega_2 t - \phi_2)$$

The total solution is then:

$$X(t) = b_1 \bar{X}_1 \cos(\omega_1 t - \phi_1) + b_2 \bar{X}_2 \cos(\omega_2 t - \phi_2)$$

Q: How do we find b_1, b_2, ϕ_1, and ϕ_2?

use the initial conditions (just like the 1 DOF case)

Example

For this system, what's the response to:

$$x_1(0) = 1\,mm, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0$$

$$x(t) = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(2t - \phi_1) + b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\sqrt{12}\,t - \phi_2)$$

and

$$\dot{x}(t) = -2b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(2t - \phi_1) - \sqrt{12}\, b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(\sqrt{12}\,t - \phi_2)$$

$$x(0) = \begin{bmatrix} 1mm \\ 0 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(-\phi_1) + b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(-\phi_2)$$

$$1mm = b_1 \cos(-\phi_1) + b_2 \cos(-\phi_2) \quad \longleftarrow \quad x_1(0) = 1mm$$

$$0mm = b_1 \cos(-\phi_1) - b_2 \cos(-\phi_2) \quad \longleftarrow \quad x_2(0) = 0mm$$

$$\dot{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -2b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(-\phi_1) - \sqrt{12}\, b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(-\phi_2)$$

$$0\,mm/s = -2b_1 \sin(-\phi_1) - \sqrt{12}\, b_2 \sin(-\phi_2) \quad \longleftarrow \quad \dot{x}_1(0) = 0\,mm/s$$

$$0\,mm/s = -2b_1 \sin(-\phi_1) + \sqrt{12}\, b_2 \sin(-\phi_2) \quad \longleftarrow \quad \dot{x}_2(0) = 0\,mm/s$$

Solving this set of equations for b_1, b_2, ϕ_1, and ϕ_2:

$b_1 = b_2 = 0.5mm$, $\phi_1 = 0$, and $\phi_2 = 0$. so the total solution is:

$$x(t) = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(2t) + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\sqrt{12}\,t)\ mm \quad \longleftarrow$$

Note: the solution in the book forgot the $\frac{1}{2}$

A Few Important Things About Eigenvectors

1. The represent the relative motion of the system. For example:

   - $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ = moving exactly together
   - $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ = moving exactly opposite
   - $\begin{bmatrix} 1 \\ -9 \end{bmatrix}$ = moving, opposite, but not equal

2. For this reason, you can think about eigenvectors as mode shapes.

3. If you excite a system at exactly a mode-shape, only that mode shows up in the response.

4. Otherwise, all modes show up with relative contributions depending on the excitation.