# MCHE 485: Mechanical Vibrations
# Spring 2020 – Multi-DOF Homework

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| Assigned: | Sunday, April 26, 2020 |
| Due: | Will not be collected, but solutions will be posted at 5pm on Wednesday, April 29 |
| Assignment: | Answer the attached problems, making sure to clearly indicate and support your answers.<br><br>*Note:*<br>These problems are from Principles of Vibration by Benson H. Tongue<br>http://amzn.com/0195142462 |
| Submission: | N/A |

# Problem 4.6

**4.6.** Find the natural frequencies and eigenvectors for the system shown in Figure P4.6. $m_1 = 15$ kg, $m_2 = 25$ kg, $k_1 = 120$ N/m, $k_2 = 200$ N/m, $k_3 = 50$ N/m.

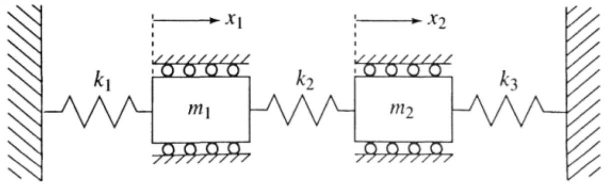



**Figure P4.6**

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2\end{bmatrix}}_{M} \underbrace{\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix}k_1+k_2 & -k_2\\ -k_2 & k_2+k_3\end{bmatrix}}_{K}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{X} = \underbrace{\begin{bmatrix}0\\0\end{bmatrix}}_{0}$$

$\leftarrow$ We set up these equations of motion in lecture.

to find eigenvectors solve:

$$\det\left(K-\omega^2 M\right)=0 \rightarrow \det\left(\begin{bmatrix}k_1+k_2-\omega^2 m_1 & -k_2\\ -k_2 & k_2+k_3-\omega^2 m_2\end{bmatrix}\right)=0$$

$$\det\left(\begin{bmatrix}320-15\omega^2 & -200\\ -200 & 250-25\omega^2\end{bmatrix}\right)=0$$

We find $\boxed{\omega_1 = 1.97\ \frac{\text{rad}}{\text{s}} \qquad \omega_2 = 5.24\ \frac{\text{rad}}{\text{s}}}$

To find the eigenvectors, plug $\omega_i$ into $\left[K-\omega_i^2 M\right]\bar{X}_i = 0$ and solve for $\bar{X}_i$

for $\omega_1$

$$\begin{bmatrix}320-15\omega^2 & -200\\ -200 & 250-25\omega^2\end{bmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix} = \begin{matrix}261.71\bar{X}_1 + (-200)\bar{X}_2 = 0\\ -200\bar{X}_1 + 152.84\bar{X}_2 = 0\end{matrix}$$

Again, these are not unique

select $\bar{X}_1 = 1$ so $\bar{X}_2 = \frac{261.71}{200}\bar{X}_1$

$\bar{X}_2 \cong 1.309$

$$\boxed{\bar{X}_1 = \begin{vmatrix}1\\1.309\end{vmatrix}}$$

for $\omega_2$

$$\begin{bmatrix}320-15\omega^2 & -200\\ -200 & 250-25\omega^2\end{bmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix} = \begin{vmatrix}0\\0\end{vmatrix} \rightarrow \begin{matrix}-91.706\bar{X}_1 - 200\bar{X}_2 = 0\\ -200\bar{X}_1 - 436.18\bar{X}_2 = 0\end{matrix}$$

select $\bar{X}_1 = 1$, then $\bar{X}_2 = \frac{-200}{436.18}\bar{X}_1 \cong -0.46\bar{X}_1$

$$\boxed{\bar{X}_2 = \begin{vmatrix}1\\-0.46\end{vmatrix}}$$

# Problem 4.7

**4.7.** Solve for the response of the system illustrated in Figure P4.6 if the initial conditions are given by

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$m_1 = 1$ kg, $m_2 = 2$ kg, $k = 3$ N/m, $k_2 = 2$ N/m, $k_3 = 4$ N/m.

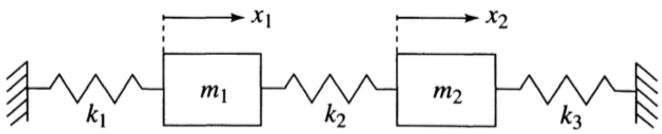
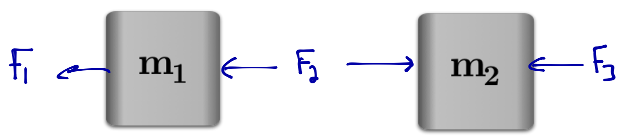



**Figure P4.7**




$F_1 = -k_1x_1 \qquad F_2 = k_2(x_1 - x_2) \qquad F_3 = k_3x_2$

$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) \rightarrow m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$

$m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3x_2 \longrightarrow m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = 0$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

To solve for the response, we first need to solve the eigenvalue problem.

$$\det(K - \omega^2 M) = 0$$

$$\det\left(\begin{bmatrix} k_1+k_2-\omega^2 m_1 & k_2 \\ k_2 & k_2+k_3-\omega^2 m_2 \end{bmatrix}\right) = 0$$

$$((k_1+k_2) - \omega^2 m_1)((k_2+k_3) - \omega^2 m_2) - k_2^2 = 0$$

$$(k_1+k_2)(k_2+k_3) - (k_2+k_3)\omega^2 m_1 - (k_1+k_2)\omega^2 m_2 + \omega^4 m_1 m_2 - k_2^2 = 0$$

Substitute values to find

$\omega^4 - 8\omega^2 + 13 = 0$ solve for $\omega^2$

$\omega_1^2 = 4 - \sqrt{3} \simeq 2.27$ and $\omega_2^2 = 4 + \sqrt{3} \simeq 5.73$

For each of these, plug into $(K - \omega_i^2 M)\bar{X} = 0$ and solve for $\bar{X}$

for $\omega_1^2 \rightarrow X = \begin{bmatrix} 0.46 \\ 0.63 \end{bmatrix}$

for $\omega_2^2 \longrightarrow \begin{bmatrix} -0.89 \\ 0.33 \end{bmatrix}$

# Problem 4.7 (cont)

The total response will be a combination of those two modes. We can write that as:

$$x(t) = b_1 \begin{bmatrix} 0.46 \\ 0.63 \end{bmatrix} \cos(\omega_1 t - \phi_1) + b_2 \begin{bmatrix} -0.89 \\ 0.33 \end{bmatrix} \cos(\omega_2 t - \phi_2)$$

Now, we need to use the initial conditions to solve for $b_1, b_2, \phi_1$ and $\phi_2$.

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.46\, b_1 \cos(-\phi_1) - 0.89\, b_2 \cos(-\phi_2) \\ 0.63\, b_1 \cos(-\phi_1) + 0.33\, b_2 \cos(-\phi_2) \end{bmatrix}$$

$$\dot{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.46\, b_1 \omega_1 \sin(-\phi_1) + 0.89\, b_2 \omega_2 \sin(-\phi_2) \\ 0.63\, b_1 \omega_1 \sin(-\phi_1) - 0.331\, b_2 \omega_2 \sin(-\phi_2) \end{bmatrix}$$

4 equations and 4 unknowns

# Problem 4.12

**4.12.** Find the equations of motion, linearize them, and find the natural frequencies and eigenvectors for the system illustrated in Figure P4.12. $m_1 = 2$ kg, $m_2 = 20$ kg, $m_3 = 1$ kg, $k_1 = 1000$ N/m, $l = 1$ m.

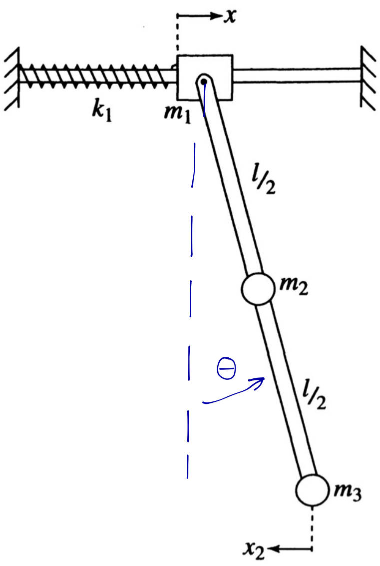
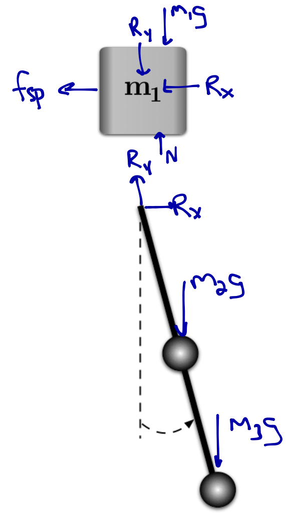



**Figure P4.12**




Find eq of motion using Lagranges Method

This system has 2DOF, choose $\bar{q} = (x, \Theta)$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x} + \frac{l}{2}\dot{\Theta}\right) + \frac{1}{2}m_3\left(\dot{x} + l\dot{\Theta}\right)$$

$$V = \frac{1}{2}kx^2 - m_2 g\, {}^{l}\!/_{2}\cos\Theta - m_3 g l\cos\Theta$$

We are assuming small angles here.

Aside: To write the velocities of $m_2$ and $m_3$, write their position, take time deriv, and linearize.

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x} + \frac{l}{2}\dot{\Theta}\right)^2 + \frac{1}{2}m_3\left(\dot{x} + l\dot{\Theta}\right)^2 - \left[\frac{1}{2}kx^2 - m_2 g\, {}^{l}\!/_{2}\cos\Theta - m_3 g l\cos\Theta\right]$$

for x

$$\frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + m_2\left(\dot{x} + \frac{l}{2}\dot{\Theta}\right) + m_3\left(\dot{x} + l\dot{\Theta}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1\ddot{x} + m_2\left(\ddot{x} + \frac{l}{2}\ddot{\Theta}\right) + m_3\left(\ddot{x} + l\ddot{\Theta}\right) = (m_1 + m_2 + m_3)\ddot{x} + \left(m_2\frac{l}{2} + m_3 l\right)\ddot{\Theta}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$(m_1 + m_2 + m_3)\ddot{x} + \left(m_2\frac{l}{2} + m_3 l\right)\ddot{\Theta} + kx = 0$$

for $\Theta$

$$\frac{\partial L}{\partial \dot{\Theta}} = m_2\left(\dot{x} + \frac{l}{2}\dot{\Theta}\right)\frac{l}{2} + m_3\left(\dot{x} + l\dot{\Theta}\right)l$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Theta}}\right) = m_2\frac{l}{2}\ddot{x} + m_2\left(\frac{l}{2}\right)^2\ddot{\Theta} + m_3 l\ddot{x} + m_3 l^2\ddot{\Theta}$$

$$\frac{\partial L}{\partial \Theta} = -m_2 g\frac{l}{2}\sin\Theta - m_3 g l\sin\Theta$$

$$\left(m_2\frac{l}{2} + m_3 l\right)\ddot{x} + \left(m_2\left(\frac{l}{2}\right)^2 + m_3 l^2\right)\ddot{\Theta} + \left(m_2\frac{l}{2} + m_3 l\right)g\sin\Theta = 0$$

linearize $\sin\Theta \approx \Theta$

$$\left(m_2\frac{l}{2} + m_3 l\right)\ddot{x} + \left(m_2\left(\frac{l}{2}\right)^2 + m_3 l^2\right)\ddot{\Theta} + \left(m_2\frac{l}{2} + m_3 l\right)g\Theta = 0$$

# Problem 4.12 (cont.)

In matrix form:

$$\underbrace{\begin{bmatrix} m_1+m_2+m_3 & m_2\frac{l}{2}+m_3 l \\ m_2\frac{l}{2}+m_3 l & m_2\left(\frac{l}{2}\right)^2+m_3 l^2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k & 0 \\ 0 & \left(m_2\frac{l}{2}+m_3 l\right)g \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

Now, solve:

1) $\det\left(K-\omega_i^2 M\right)=0$ for eigenvalues $\omega_i^2$

2) For each $\omega_i^2$, solve $\left[K-\omega_i^2 M\right]X_i=0$ for eigenvector $X_i$

# Problem 4.35

**4.35.** Determine the eigenvalues and eigenvectors of the following system: $a = b = .5$ m, $m_1 = m_2 = 1$ kg, $k = 1$ N/m, $g = 5$ m/s$^2$ (Obviously the system, shown in Figure P4.35, isn't on Earth.)

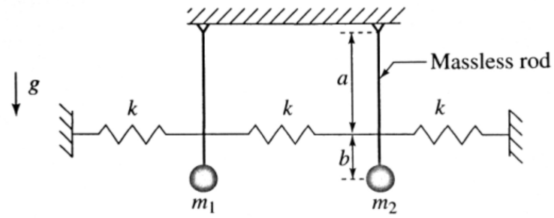
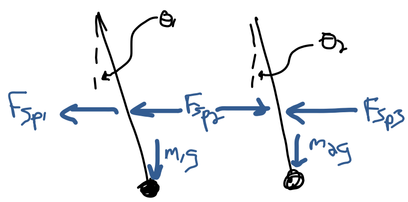



Figure P4.35

## FBD




Assuming small angles about $\theta_1 = \theta_2 = 0$

$F_{sp1} = -ka\theta_1$ $\qquad$ $F_{sp3} = -ka\theta_3$

$F_{sp2} = ka(\theta_1 - \theta_2)$ ← the sign of this will change for $\theta_1$ and $\theta_2$ eq

### $\theta_1$ equation

$$m_1(a+b)^2\ddot{\theta}_1 = -ka^2\theta_1 - ka^2(\theta_1 - \theta_2) - mg(a+b)\theta_1$$

$$m_1(a+b)^2\ddot{\theta}_1 + \left(2ka^2 + m_1 g(a+b)\right)\theta_1 - 2ka^2\theta_2 = 0$$

### $\theta_2$ equation

$$m_2(a+b)^2\ddot{\theta}_2 = ka^2(\theta_1 - \theta_2) - ka^2\theta_2 - m_2 g(a+b)\theta_2$$

$$m_2(a+b)^2\ddot{\theta}_2 - ka^2\theta_1 + \left(2ka^2 + m_2 g(a+b)\right)\theta_1 = 0$$

In matrix form:

$$\underbrace{\begin{bmatrix} m_1(a+b)^2 & 0 \\ 0 & m_2(a+b)^2 \end{bmatrix}}_{M}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2ka^2 + m_1 g(a+b) & -ka^2 \\ -ka^2 & 2ka^2 + m_2 g(a+b) \end{bmatrix}}_{K}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvalue problem is then:

$$\det\left(\left[K - \omega^2 M\right]\right) = 0$$

## Problem 4.35 (cont.)

$$\underbrace{\begin{bmatrix} m_1(a+b)^2 & 0 \\ 0 & m_2(a+b)^2 \end{bmatrix}}_{M}\begin{vmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{vmatrix} + \underbrace{\begin{bmatrix} 2ka^2+m_1g(a+b) & -ka^2 \\ -ka^2 & 2ka^2+m_2g(a+b) \end{bmatrix}}_{K}\begin{vmatrix}\theta_1\\ \theta_2\end{vmatrix} = \begin{vmatrix}0\\ 0\end{vmatrix}$$

The eigenvalue problem is then:

$$\det\left([K-\omega^2 M]\right)=0$$

Filling in the values $m_1=m_2=1$ $a=b=0.5$ $g=5.0$ $k=1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{vmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{vmatrix} + \begin{bmatrix} 5.5 & -0.25 \\ -0.25 & 5.5 \end{bmatrix}\begin{vmatrix}\theta_1\\ \theta_2\end{vmatrix} = \begin{vmatrix}0\\ 0\end{vmatrix}$$

$$\det\left(\begin{bmatrix} 5.5-\omega^2 & -0.25 \\ -0.25 & 5.5-\omega^2 \end{bmatrix}\right)=0$$

$$(5.5-\omega^2)^2-0.0625=0$$

$$30.25-11\omega^2+\omega^4-0.0625=0$$

$$\omega^4-11\omega^2+30.188=0$$

$$\omega^2=\frac{11\pm\sqrt{121-4(1)(30.188)}}{2}=\frac{11\pm\sqrt{0.25}}{2}=\frac{11\pm0.5}{2}$$

$$\boxed{\omega_1^2=5.25 \quad \omega_2^2=5.75}$$

Eigenvector problem

for $\omega_1^2$ — $[K-\omega_1^2M]|\bar{X}_1|=0 \rightarrow \begin{bmatrix} 5.5-5.25 & -0.25 \\ -0.25 & 5.5-5.25 \end{bmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix}=0 \rightarrow$ $0.25\bar{X}_1-0.25\bar{X}_2=0$, $-0.25\bar{X}_1+0.25\bar{X}_2=0$

so $\boxed{\bar{X}_1=\begin{bmatrix}1\\1\end{bmatrix}}$

for $\omega_2^2$ — $[K-\omega_2^2M]|\bar{X}_2|=0 \rightarrow \begin{bmatrix} 5.5-5.75 & -0.25 \\ -0.25 & 5.5-5.75 \end{bmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix}=0 \rightarrow \begin{bmatrix} -0.25 & -0.25 \\ -0.25 & -0.25 \end{bmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix}=0$

so $\boxed{\bar{X}_2=\begin{vmatrix}1\\-1\end{vmatrix}}$

# Problem 4.60

**4.60.** At what frequency of forcing will the mass $m_2$ in Figure P4.60 be stationary? What will the forcing amplitude be equal to if $m_1$ is limited to an excursion of 3 mm? $m_1 = .4$ kg, $m_2 = .8$ kg, $k = 3000$ N/m.

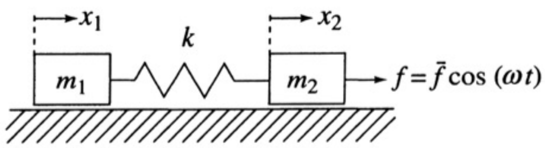
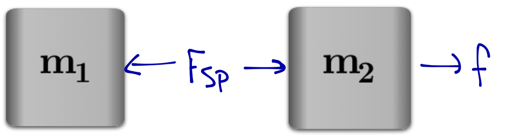



**Figure P4.60**




$$m_1\ddot{x}_1 = -F_{sp} = -k(x_1 - x_2) \rightarrow m_1\ddot{x}_1 + kx_1 - kx_2 = 0$$

$$m_2\ddot{x}_2 = F_{sp} + f = k(x_1 - x_2) + f \rightarrow m_2\ddot{x}_2 - kx_1 + kx_2 = f$$

In matrix form:

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ f \end{bmatrix}}_{F}$$

To solve for the response, solve:

$$\bar{X} = [K - \omega^2 M]^{-1}\bar{F}$$

$$= \begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \bar{f} \end{bmatrix} = \frac{1}{\det(K - \omega^2 M)}\begin{bmatrix} k - \omega^2 m_2 & k \\ k & k - \omega^2 m_1 \end{bmatrix}\begin{bmatrix} 0 \\ \bar{f} \end{bmatrix}$$

So:

$$\bar{X} = \frac{1}{\det(K - \omega^2 M)}\begin{bmatrix} k\bar{f} \\ (k - \omega^2 m_1)\bar{f} \end{bmatrix} \qquad \bar{X}_1 = \frac{k\bar{f}}{\det(K - \omega^2 M)} \qquad \bar{X}_2 = \frac{(k - \omega^2 m_1)\bar{f}}{\det(K - \omega^2 M)}$$

← amplitude = 0 when the numerator = 0, $(k - \omega^2 m_1)\bar{f} = 0$

$$\omega^2 = \frac{k}{m_1} \rightarrow \boxed{\omega = \sqrt{\frac{k}{m_1}}}$$

Examine the response @ $\omega = \sqrt{\frac{k}{m_1}}$:

$$\det\left(K - \left(\frac{k}{m_1}\right)M\right) = -9{,}000{,}000$$

for $x_1 = 3\text{mm} = \frac{k\bar{f}}{-9{,}000{,}000}$ $\quad \boxed{\bar{f} = (0.003)(9{,}000{,}000)\left(\frac{1}{3000}\right) = 9\text{ N}}$

# Problem 4.75

**4.75.** Consider the system shown in Figure P4.52*b*. Let $m_1 = 1$ kg, $m_2 = 5$ kg, $k_1 = 100$ N/m, $k_2 = 200$ N/m, and $k_3 = 50$ N/m; $f_1(t) = 1.8318\cos(10t)$ and $f_2(t) = -1.3145\cos(10t)$ (forces given in newtons). Put the system into normal form and show that under the given forcing, the system responds in the second mode (the eigenvector associated with the highest frequency).

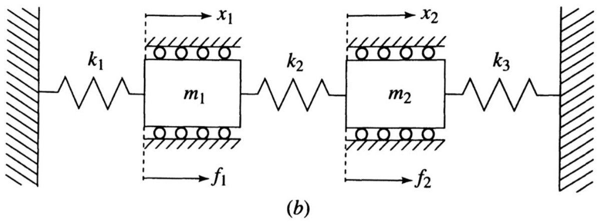



**Figure P4.52**

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{F}$$

$$M\ddot{X} + KX = F$$

1) Solve $\det(K - \omega^2 M) = 0$ for the two eigenvalues

2) For each eigenvalue $\omega_i^2$, solve $[K - \omega_i^2 M]X_i = 0$ for the corresponding eigenvector $X_i$

3) Normalize these eigenvectors by $\tilde{X}_i = \frac{1}{X_i^T M X_i} X_i$

4) Form $U = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \end{bmatrix}$, using the normalized eigenvectors as columns.

5) To put into Normal form:

$$U^T M U \ddot{H} + U^T K U H = U^T F$$

Solving for this in the IPython Notebook, we find:

$$\ddot{H} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} H = \underbrace{\begin{bmatrix} 0 \\ -1.924 \end{bmatrix}}_{\text{This shows that only the 2nd mode is excited.}} \cos(10t)$$