______ **Problem 2 – 25 Points**

The system in Figure 2 is a common model used for rotating machinery. It consists of a mass, m_1, connected to ground through a spring k, and a damper, c. A second mass, m_2, is offset from center by eccentricity, e, and is rotating at an angular velocity of ω. There is a secondary mass, m_3, and spring, k_2, that will be added in for parts d.–g. of this problem.

Ignoring the secondary, k_2–m_3, subsystem:

a. Write the equations of motion describing the motion of m_1.

b. What is the natural frequency?

c. Sketch the approximate frequency response. Be sure to indicate:

   i. Magnitude as ω approaches 0.

   ii. Magnitude as ω approaches infinity.

   iii. Magnitude when ω equals the natural frequency of the system, ω_n.

Now, the secondary subsystem consisting of k_2 and m_2 is added.

d. How should this subsystem be designed such that m_1 remains stationary, or nearly stationary, over a range of input frequencies near the natural frequency of the original system. Give values for k_2 and m_3 in terms of k_1, m_1, m_2, and e.

e. What is the fundamental compromise in the design for part d. ?

f. Write the equations of motion for the new system, including the secondary, k_2–m_3, subsystem.

g. Sketch the approximate frequency response for x and x_3 for this new system.

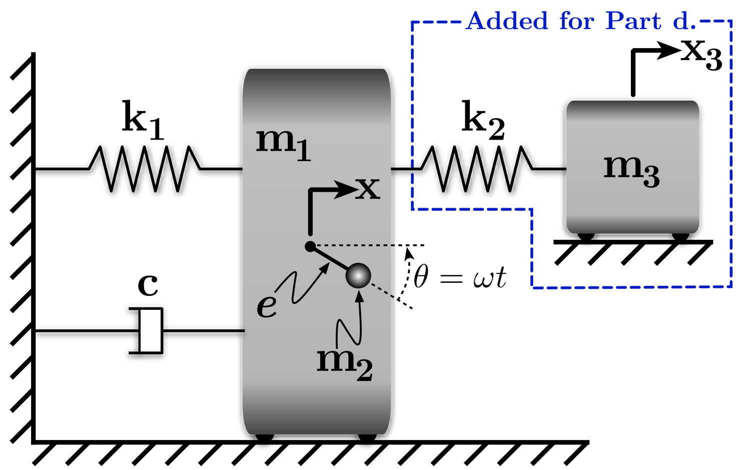


Figure 2: A Rotating Imbalance System

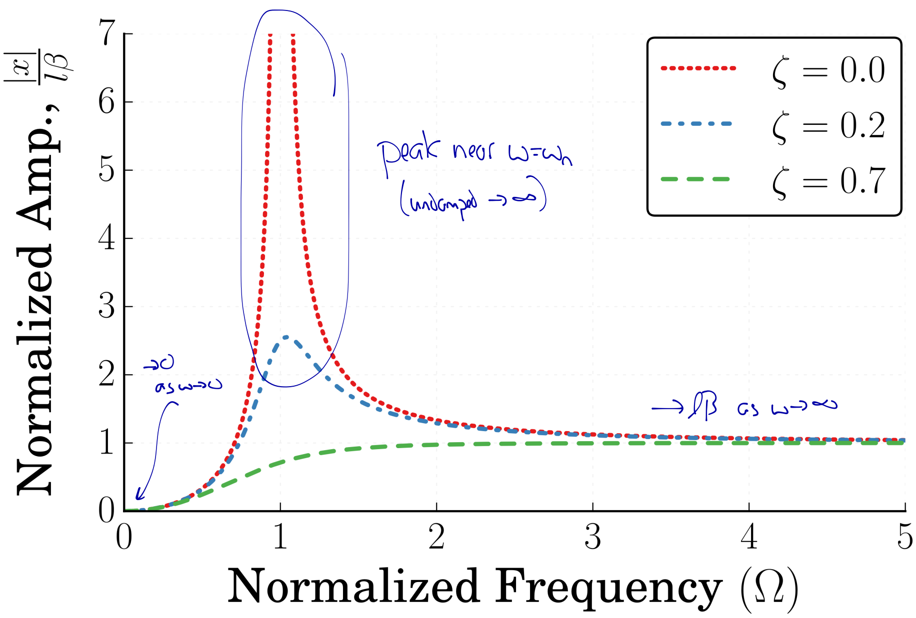
Problem 2

a. This is just a "standard" rotating imbalance system:

$$\boxed{(m_1+m_2)\ddot{x} + c\dot{x} + kx = m_2 e\omega^2\cos\omega t \quad \text{or} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = e\beta\omega^2\cos\omega t \quad \text{where} \quad 2\zeta\omega_n = \frac{c}{m_1+m_2} \quad \omega_n^2 = \frac{k_1}{m_1+m_2} \quad \text{and} \quad \beta = \frac{m_2}{m_1+m_2}}$$

b. $\boxed{\omega_n = \sqrt{\frac{k_1}{m_1+m_2}}}$

c.



d. In order for m1 to remain stationary, the k2--m3 subsystem would have to act as a vibration absorber. For this to happen the natural frequency of that subsystem should match the desired "zero" frequency for m1. Here, that is the natural frequency of the original system, so:

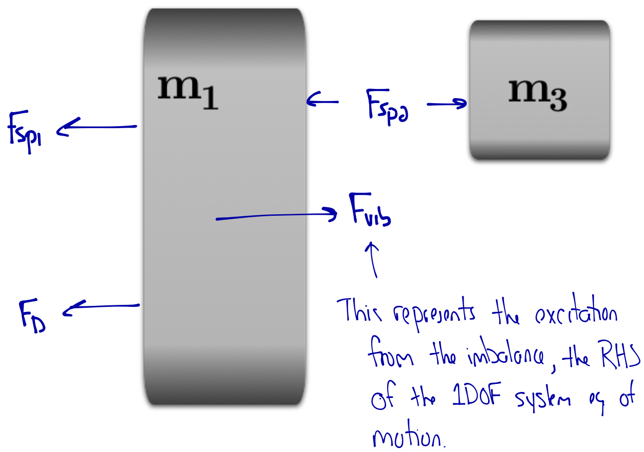
Input freq $\omega = \sqrt{\frac{k_1}{m_1+m_2}}$ so we want $\sqrt{\frac{k_2}{m_3}} = \omega = \sqrt{\frac{k_1}{m_1+m_2}} \longrightarrow \boxed{\frac{k_2}{m_3} = \frac{k_1}{m_1+m_2}}$

e. • Large absorber mass leads to:
- low amplitude absorber mass motion
- robustness around the desired freq.
- at the cost of increased mass (and likely volume)

• Small 2nd mass leads to:
- higher amplitude absorber mass motion
- less robustness to changes in frequency
- benefit is a lighter/smaller system

Problem 2 (cont.)

f.



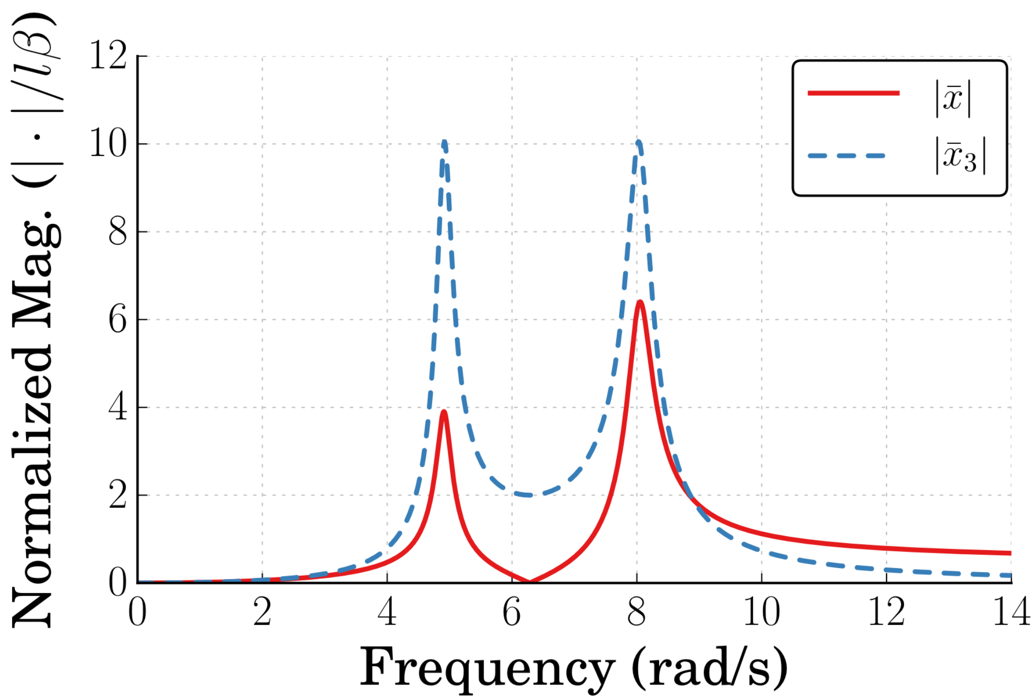
The main system equation is nearly the same, only with the addition of the second spring force.

$$\boxed{\begin{aligned} (m_1+m_2)\ddot{x} + c\dot{x} + (k_1+k_2)x - k_2 x_3 &= m_2 l \omega^2 \cos\omega t \\ m_3 \ddot{x}_3 - k_2 x + k_2 x_3 &= 0 \end{aligned}}$$

In matrix form:

$$\boxed{\begin{bmatrix} m_1+m_2 & 0 \\ 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x \\ x_3 \end{bmatrix} = \begin{bmatrix} m_2 l \omega^2 \cos\omega t \\ 0 \end{bmatrix}}$$

g.



______ **Problem 3 – 25 Points**

The system in Figure 3 is a mass-spring system moving along the ground. Figure 4 is a similar system with friction. The coefficient of friction between the mass and ground in this system is μ.

a. Write the equations of motion for these two systems.

b. For both systems, plot the response, $x(t)$, to initial conditions:

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = -v_0$$

For the system in Figure 3, plot the response for a damping ratio $\zeta = 0.1$. For the system in Figure 4, use a coefficient of friction that would result in approximately the same settling time. Be sure to clearly label the axes and differentiate between the responses. Indicate what is different between the responses for two energy dissipation models.

A coworker suggests attaching the system in Figure 5 to the main system mass. He claims that this proposed sensor can measure acceleration.

c. For the proposed sensor in Figure 5, write the transfer function between the sensor motion, $y(t)$, and the measurement, $x(t)$.

d. Explain how the sensor can be used to measure acceleration. (*Hint*: Using the transfer function from part c., write the relationship between $x(t)$ and $\ddot{y}(t)$.)

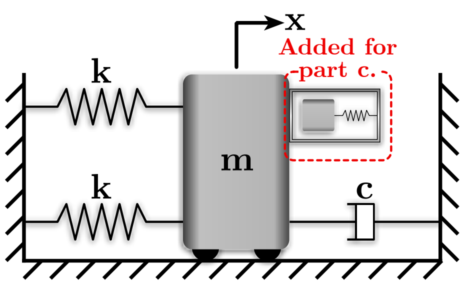


Figure 3: A Mass-Spring-Damper System

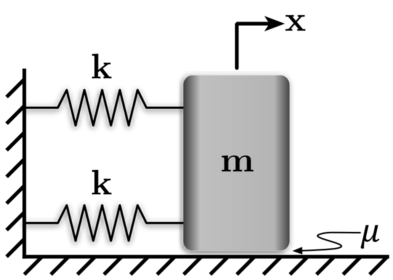


Figure 4: A Mass-Spring System with Friction

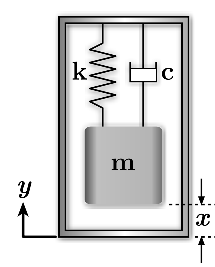
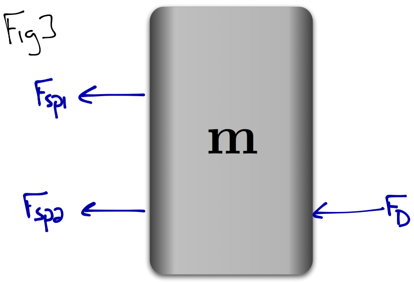


Figure 5: The Proposed Sensor

Problem 3

a. Fig3



For Fig 4

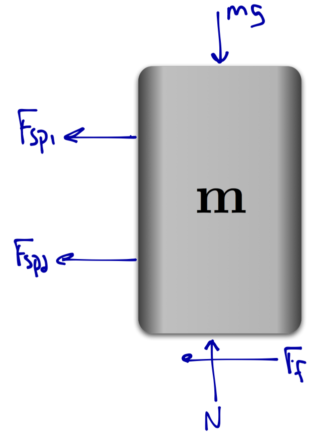


Fig 3:

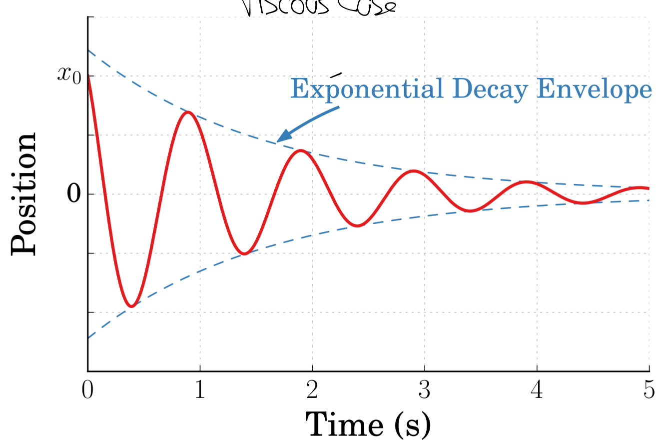
$$m\ddot{x} = -F_{sp1} - F_{sp2} - F_D$$
$$m\ddot{x} = -kx - kx - c\dot{x}$$
$$\boxed{m\ddot{x} + c\dot{x} + 2kx = 0}$$

Fig 4:

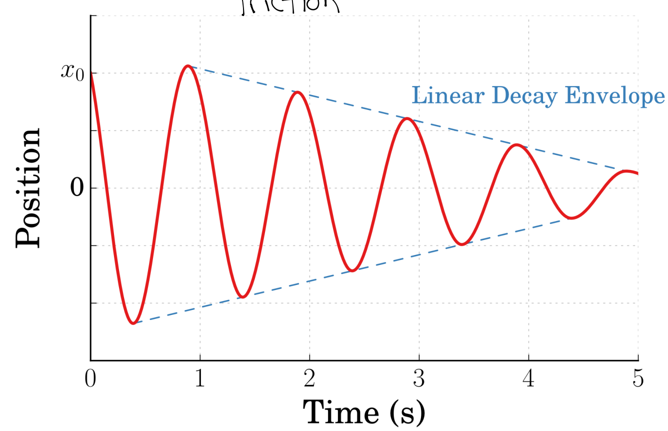
$$m\ddot{x} = -F_{sp1} - F_{sp2} - F_f \qquad N = mg$$
$$m\ddot{x} = -kx - kx - \mu N \,\text{sgn}(\dot{x})$$
$$\boxed{m\ddot{x} + \mu mg \,\text{sgn}(\dot{x}) + 2kx = 0}$$

b.

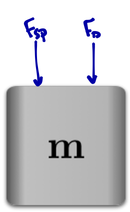
Viscous Case



Friction



c.



$$m(\ddot{x}+\ddot{y}) = -F_{sp} - F_D$$
$$= -kx - c\dot{x}$$

Assume $y(t) = \bar{y}e^{i\omega t}$
and $x(t) = \bar{x}e^{i\omega t}$

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{y}$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -\ddot{y}$$

plug into the eq of motion

$$(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2)\bar{x}\cancel{e^{i\omega t}} = \omega^2\bar{y}\cancel{e^{i\omega t}}$$

$$\boxed{\frac{\bar{X}}{\bar{Y}} = \frac{\omega^2}{(\omega_n^2-\omega^2) + 2i\zeta\omega\omega_n}}$$

d. So, $x(t) = \frac{\omega^2}{(\omega_n^2-\omega^2)+2i\zeta\omega\omega_n}\bar{y}e^{i\omega t}$ ← This is $\ddot{y}$ → $x(t) = \frac{-\ddot{y}(t)}{(\omega_n^2-\omega^2)+2i\zeta\omega\omega_n}$

If $\omega \ll \omega_n$ (excitation freq is much lower than sensor natural freq)

$$x(t) \approx \frac{-1}{\omega_n^2}\ddot{y}(t) \longrightarrow \ddot{y}(t) = -\omega_n^2 x(t)$$

So, to measure accel, measure $x(t)$ and multiply by $-\omega_n^2$

______ **Problem 1 – 15 Points**

The system in Figure 1 consists of mass, m, excited by input, $y(t)$, through springs of spring constant, k. There is also a damper, c, between the mass and ground.

a. Write the equations of motion for this system.

b. What is the natural frequency?

c. Assuming $y(t) = 0$, plot the response, $x(t)$, to initial conditions:

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses.

d. Write the response, $x(t)$, to these same initial conditions.

e. Now, assume $y(t) = \bar{y}e^{i\omega t}$. Write the transfer function from the amplitude of the input, $\bar{y}$, to the amplitude of the response.

f. Sketch the approximate frequency response (both magnitude and phase) for $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. For each damping ratio, be sure to indicate:

   i. Amplitude and phase as ω approaches 0.

   ii. Amplitude and phase as ω approaches infinity.

   iii. Amplitude when ω equals the natural frequency of the system, ω_n.

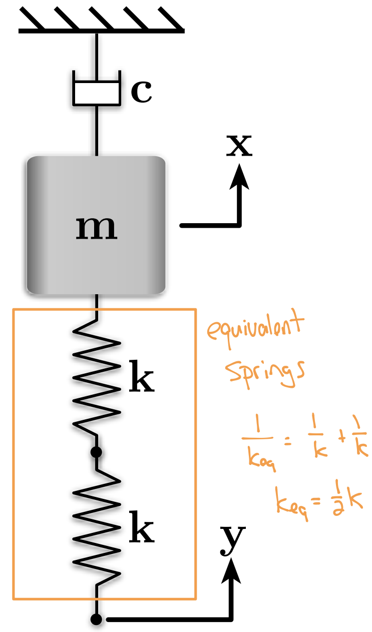
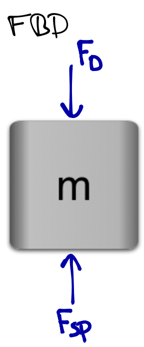


Figure 1: A Mass-Spring-Damper System

a)



$m\ddot{x} = F_{sp} - F_D$ $\qquad F_{sp} = k_{eq}(y - x)$

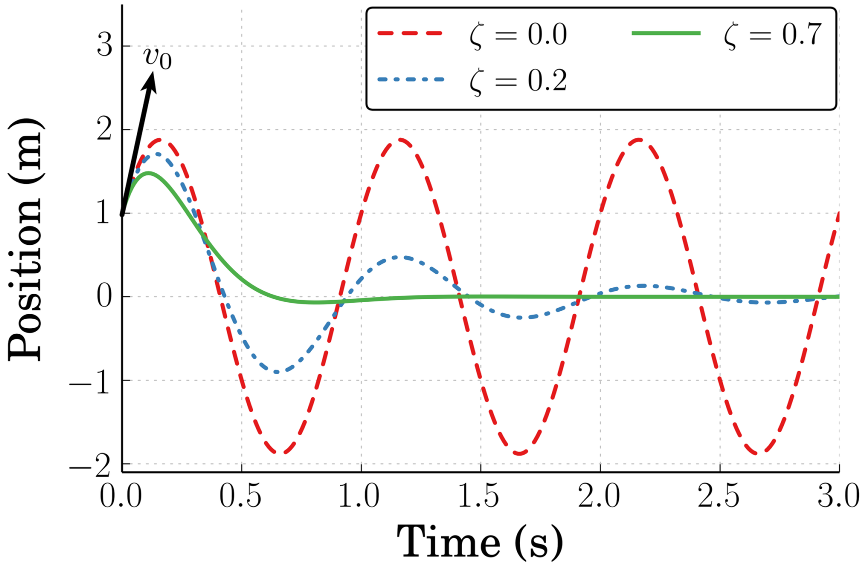
$m\ddot{x} = k_{eq}(y-x) - c\dot{x}$ $\qquad F_D = c\dot{x}$

$$\boxed{m\ddot{x} + c\dot{x} + k_{eq}x = k_{eq}y \quad \text{or} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 y \quad \text{where } 2\zeta\omega_n = \frac{c}{m} \text{ and } \omega_n^2 = \frac{k_{eq}}{m}}$$

b) The natural frequency is $\boxed{\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2m}}}$

Problem 1 (cont.)

c)



d) We can write the response as:

$$x(t) = e^{-\zeta\omega_n t}\left(b_1 \cos\omega_d t + b_2 \sin\omega_d t\right)$$

To use this equation, we need to solve for b_1 and b_2 using the initial conditions. Solving the equation for generic initial velocity, $\dot{x} = v_0$, and a generic initial displacement, $x = x_0$, we find:

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + v_0}{\omega_d}\sin\omega_d t\right)$$

e) This is just a seismically-excited mass-spring system. The equation was on the equation sheet.

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 y$

$y(t) = \bar{Y}e^{i\omega t}$ so assume $x(t) = \bar{X}e^{i\omega t}$

$\dot{x}(t) = i\omega\bar{X}e^{i\omega t}$

$\ddot{x}(t) = -\omega^2 e^{i\omega t}$

plug into the eq of motion

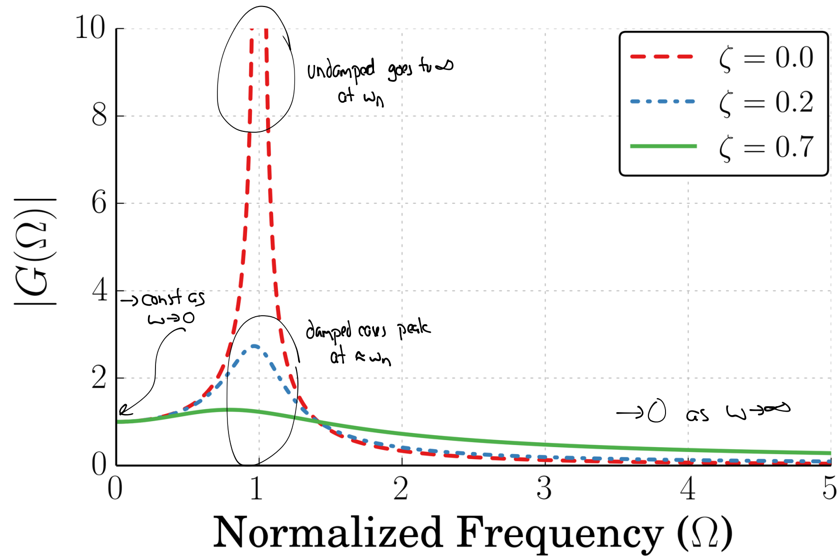
⋮

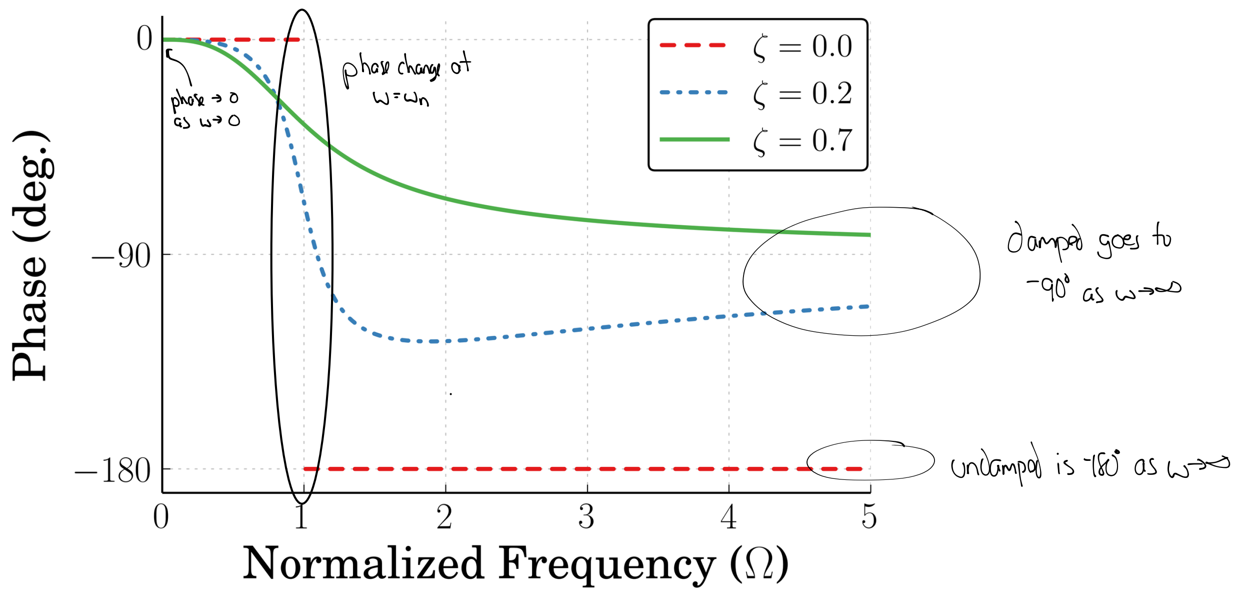
$(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2)\bar{X}\cancel{e^{i\omega t}} = \omega_n^2\bar{Y}\cancel{e^{i\omega t}}$

$$\boxed{\frac{\bar{X}}{\bar{Y}} = \frac{\omega_n^2}{(\omega_n^2-\omega^2)+(2i\zeta\omega\omega_n)}}$$

Problem 1 (cont.)

f)





______ **Problem 2 – 15 Points**

The system in Figure 2 consists of a thin rod of mass m and length l. It is connected to ground via a damper attached at distance l_1 from the rod's pivot point and a spring, k, at a distance of l_2 from the pivot. There is a pure torque, τ, acting on the rod. You can assume that the spring is at its equilibrium when $\theta = 0$.

a. Write the equations of motion for this system.
b. What is the natural frequency?
c. Assuming $\tau(t) = \bar{\tau}e^{i\omega t}$, write the transfer function from the amplitude of the torque, $\bar{\tau}$, to the amplitude of the response.
d. Sketch the approximate frequency response (both magnitude and phase) for $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. For each damping ratio, be sure to indicate:
   i. Amplitude and phase as ω approaches 0.
   ii. Amplitude and phase as ω approaches infinity.
   iii. Amplitude and phase when ω equals the natural frequency of the system, ω_n.

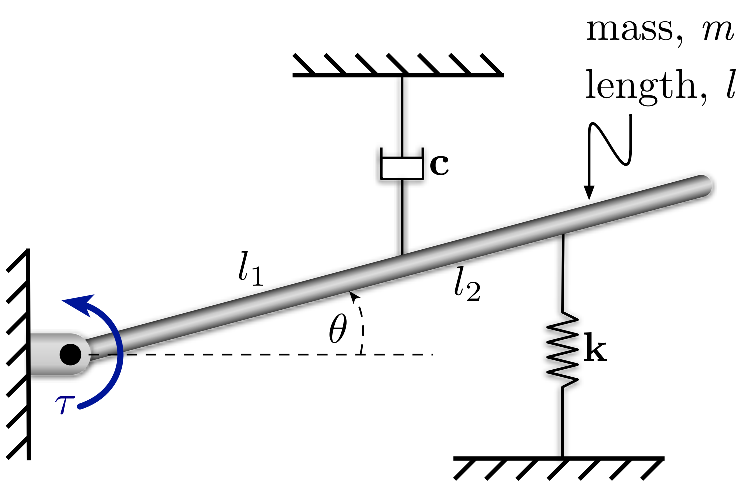
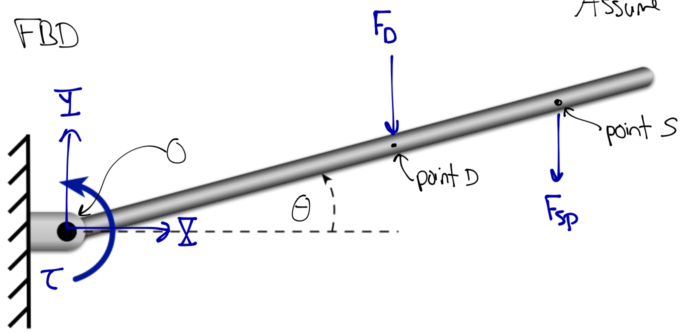


Figure 2: A Thin Rod with a Direct Torque Input

Problem 2 (cont.)

a) FBD



Assume $\theta \approx 0$,

$F_D = c l_1 \dot{\theta}$

$F_{sp} = k l_2 \theta$

Sum moments about point O

$$\sum \bar{M}_O = \tau \bar{K} + \left(\bar{r}_{D/O} \times -F_D \bar{J}\right) + \left(\bar{r}_{S/O} \times -F_{sp} \bar{J}\right)$$

$$= \tau \bar{K} + \left(l_1 \cos\theta \bar{I} + l_2 \sin\theta \bar{J} \times -F_D \bar{J}\right) + \left(l_2 \cos\theta \bar{I} + l_2 \sin\theta \bar{J} \times -F_{sp} \bar{J}\right)$$

$$= \tau \bar{K} + \left(-F_D l_1 \cos\theta \bar{K}\right) + \left(-F_{sp} l_2 \cos\theta \bar{K}\right)$$

$$= \tau \bar{K} + \left(-c_1 l_1^2 \dot{\theta} \bar{K}\right) + \left(-k l_2^2 \theta \bar{K}\right)$$

$$I\ddot{\theta} = -kl^2\theta - cl^2\dot{\theta} + \tau$$

$$\boxed{I\ddot{\theta} + c l_1^2 \dot{\theta} + k l_2^2 \theta = \tau \quad \text{where } I = \tfrac{1}{3} m l^2}$$

or

$$\boxed{\ddot{\theta} + 2\zeta\omega_n \dot{\theta} + \omega_n^2 \theta = \tau / I \quad \text{where } 2\zeta\omega_n = \frac{c l_1^2}{I} \text{ and } \omega_n^2 = \frac{k l_2^2}{I}}$$

b) The natural frequency is $\boxed{\omega_n = \sqrt{\frac{k l_2^2}{I}} = \sqrt{\frac{k l_2^2}{\frac{1}{3} m l^2}} = \sqrt{\frac{3 k l_2^2}{m l^2}}}$

c) This is a direct-force system so we can use that form of the transfer function:

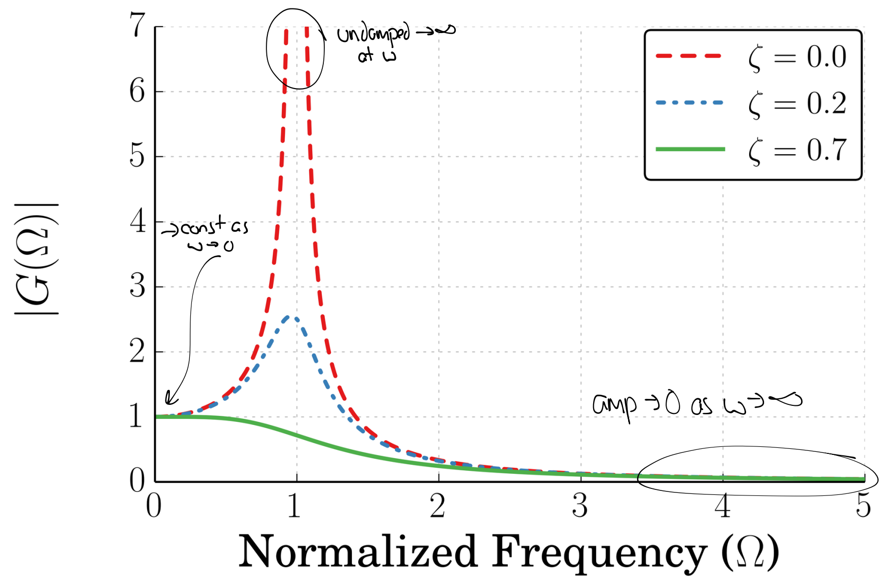
$$\frac{\bar{\theta}}{\bar{\tau}} = \frac{1}{I} \frac{1}{(\omega_n^2 - \omega^2) + (2i\zeta\omega\omega_n)}$$

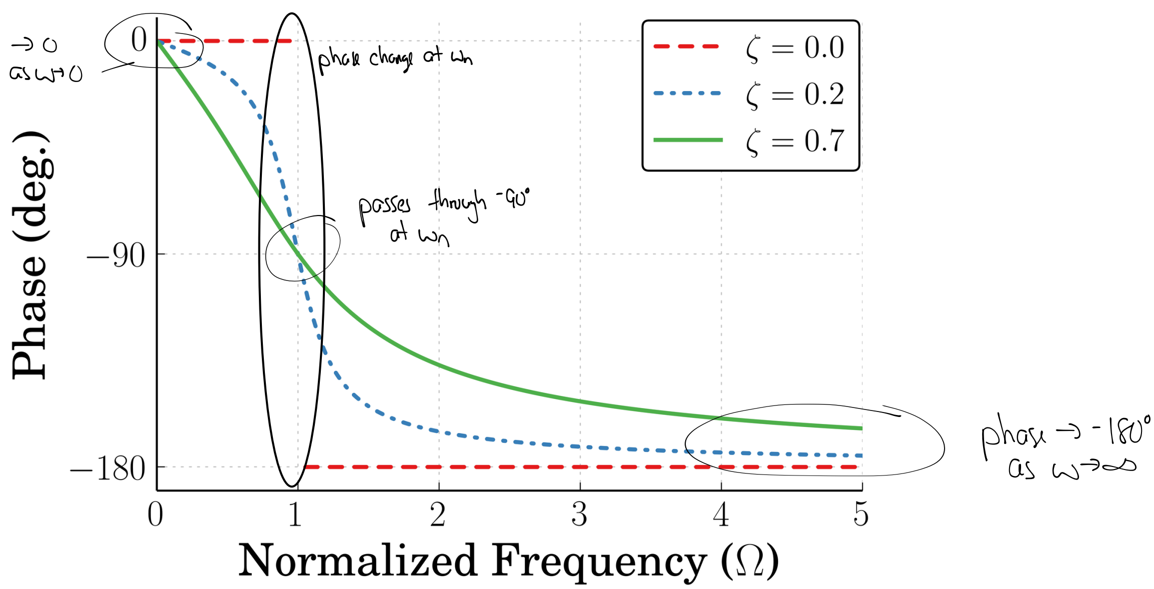
We can use this to write the response as:

$$\theta(t) = \left[\frac{\bar{\tau}}{I} \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}}\right] \cos(\omega t - \phi), \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

Problem 2 (cont.)

d)





______ **Problem 1 – 25 Points**

The system in Figure 1 consists of a mass, m, connected to ground through through spring k and to an input, y, via damper c.

a. Write the equations of motion for this system.

b. What is the natural frequency?

c. Assume $y(t) = 0$ (*i.e.* it acts like another ground connection with respect to the mass). Write the response, $x(t)$, to initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$.

d. Plot the response, $x(t)$, to the same initial conditions for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

e. Now, assume a harmonic input in $y(t)$. Write the transfer function from the amplitude of the input to the amplitude of the output.

f. Assuming $y(t) = \bar{y}\cos\omega t$, write the time response, $x(t)$.

g. Set up the solution procedure to determine the time response, $x(t)$, to the *velocity* input, $\dot{y}(t)$, shown in Figure 2. Define as much as possible based on the information you have been given. If there are integrals needed, you do not need to solve them, but do set the problem up such that it could be passed to a calculus student to do so. If terms in the integration will be zero, please be nice to the calculus student and indicate so.

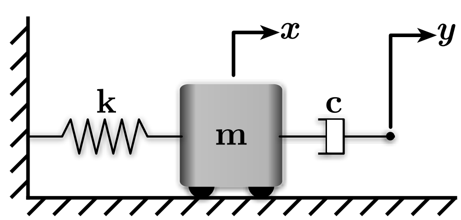


Figure 1: Mass-Spring-Damper System

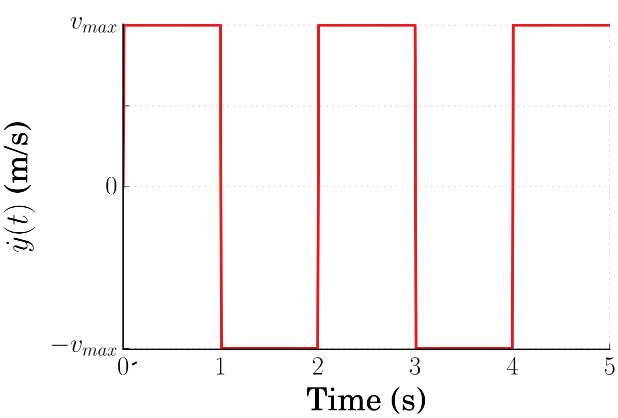
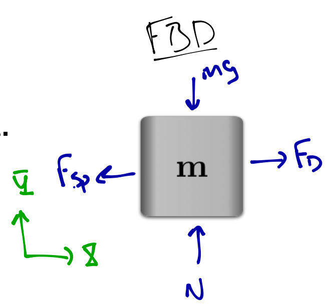


Figure 2: Velocity Command

a.



$m\ddot{x} = -F_{sp} + F_D = -kx + c(\dot{y}-\dot{x})$

$$\boxed{m\ddot{x} + c\dot{x} + kx = c\dot{y} \quad \text{or} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} \quad \text{where } 2\zeta\omega_n = \frac{c}{m} \text{ and } \omega_n^2 = \sqrt{k/m}}$$

Problem 1 (cont.)

b. $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$

c. If $y = \dot{y} = 0$, the equation of motion is

$m\ddot{x} + c\dot{x} + kx = 0$ and the system is just in free vibration

so

$$x(t) = e^{-\zeta\omega_n t}\left(a\cos\omega_d t + b\sin\omega_d t\right)$$

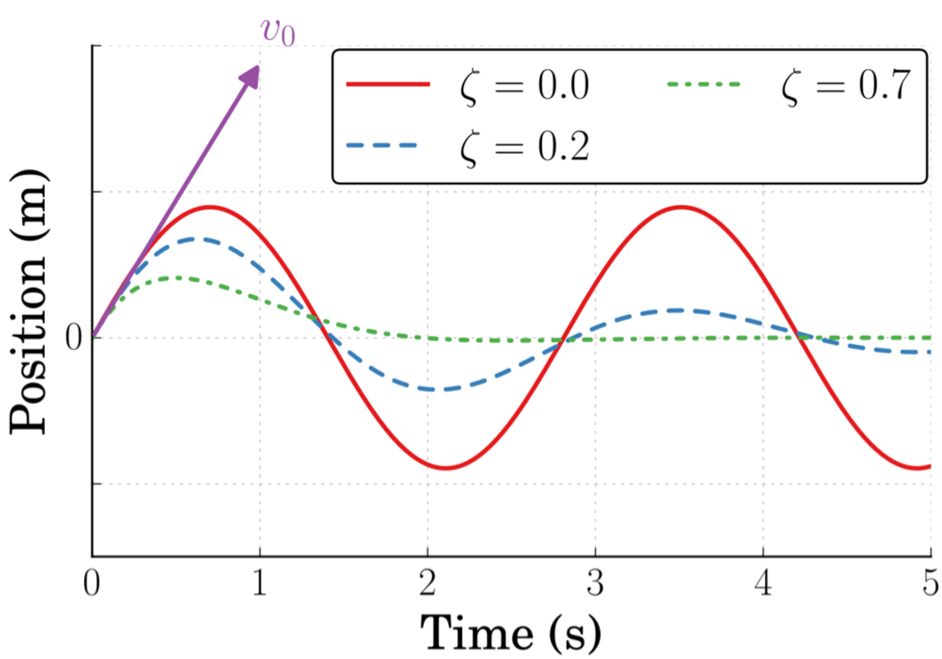
$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}\left(a\cos\omega_d t + b\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(-a\omega_d\sin\omega_d t + b\omega_d\cos\omega_d t\right)$$

$$x(0) = 0 = a$$

$$\dot{x}(0) = v_0 = -\zeta\omega_n\cancel{(a)}^{0} + (b\omega_d) \rightarrow b = \frac{v_0}{\omega_d}$$

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left(\frac{v_0}{\omega_d}\sin\omega_d t\right)}$$

d.



Problem 1 (cont.)

e. If $y(t)$ is a pure harmonic, assume $y(t)=\bar{y}e^{i\omega t}$

Assume $x(t)=\bar{x}e^{i\omega t}$ to match the form of the input

Sub into the eq. of motion

$$\left(-\omega^2+2i\zeta\omega\omega_n+\omega_n^2\right)\bar{x}\cancel{e^{i\omega t}} = 2i\zeta\omega\omega_n\bar{y}\cancel{e^{i\omega t}}$$

$$\boxed{\frac{\bar{X}}{\bar{Y}} = \frac{2i\zeta\omega\omega_n}{(\omega_n^2-\omega^2)+(2i\zeta\omega\omega_n)}}$$

or

$$\frac{\bar{X}}{\bar{Y}} = \frac{2\zeta\omega\omega_n}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{i(\phi_1-\phi_2)}$$

where $\phi_1 = \pi/2$ and

$$\phi_2 = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)$$

f. $$\boxed{x(t) = \frac{2i\zeta\omega\omega_n}{(\omega_n^2-\omega^2)+(2i\zeta\omega\omega_n)}\bar{y}e^{i\omega t}}$$

If $y(t)$ was defined as a cosine, take the real part of the above.
a sine, take the imaginary part of the above.

g. The input is repeating, so we can use Fourier Analysis.

$\tau_0 = 2s$ (signal repeats every 2s) $\rightarrow$ $\omega_0 = \frac{2\pi}{2s} = \pi \frac{rad}{s}$

$$\dot{y}(t) = \begin{cases} V_{max} & 0\le t<1s \quad (\text{also } 0\le t<\pi/\omega_0) \\ -V_{max} & 1\le t<2s \quad (\text{also } \pi/\omega_0\le t<2\pi/\omega_0) \end{cases}$$

Now, just plug in $f(t)=\dot{y}(t)$ and ω_0 into the Fourier Analysis Formula

$$f(t) = \cancel{\sum_{n=0}^{\infty} a_n\cos(n\omega_0 t)}^{\,0} + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)$$

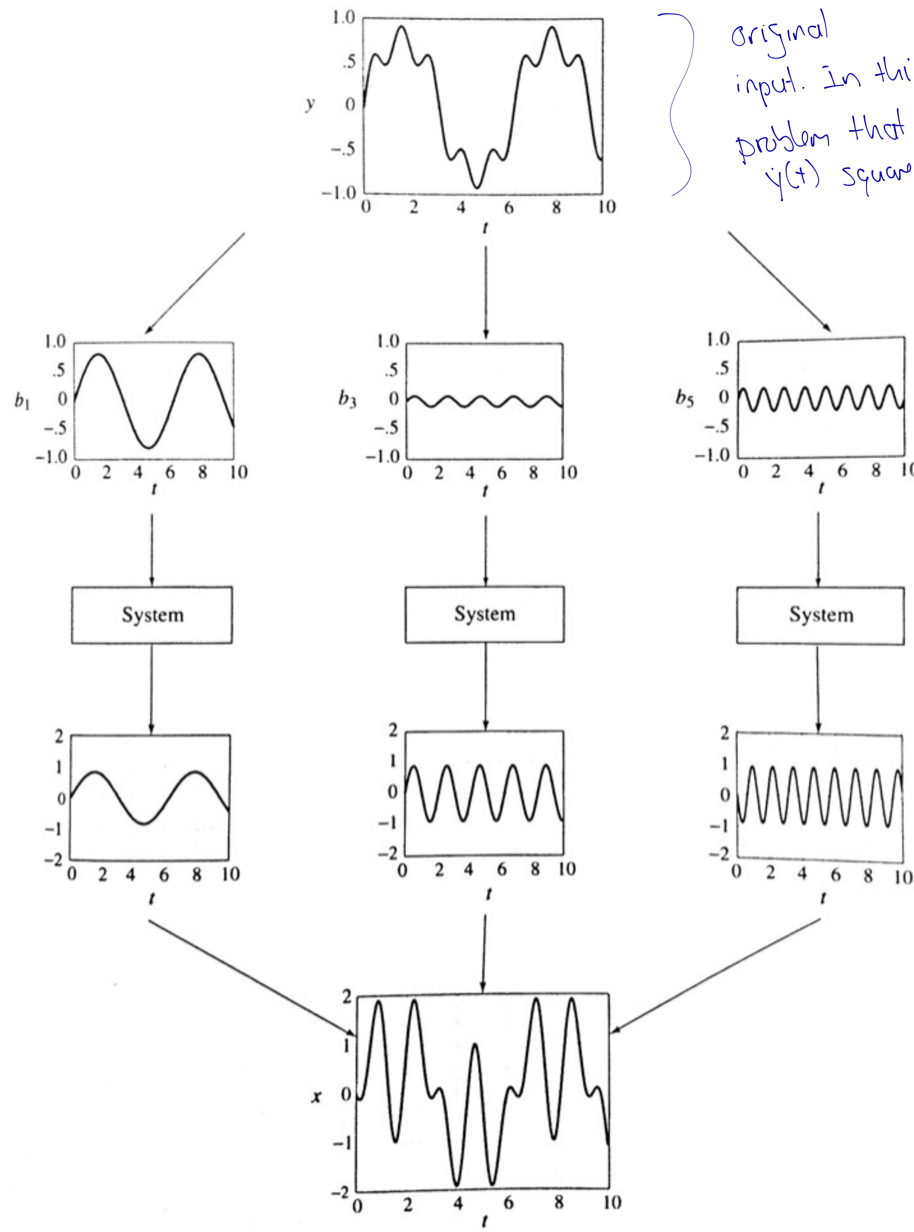
$$\cancel{a_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\cos(n\omega_0 t)dt}$$ All cos() terms = 0 because $y(t)$ is odd

$$b_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)dt$$ $\leftarrow$ Just solve for b_n terms

$$\cancel{a_0 = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)dt}$$ $a_0 = 0$ because $\dot{y}(t)$ is centered around 0

Problem 1 (cont.)

g (cont.). Once $y(t)$ is expanded using Fourier Analysis the response can be found by finding the response to each component of the resulting sum of sine inputs. This process is nicely summarized by Figure 3.3 from the book.



original input. In this problem that is the $y(t)$ square wave

Fourier Analysis outlined on the previous page

Determine the response to each component that results from the Fourier Analysis

Sum the individual responses to get the total response

Figure 3.3 Schematic of how Fourier analysis works