# MCHE 485: Mechanical Vibrations

Spring 2020 – Final Exam

Due: 11:59pm, Tuesday, May 5

**Name:** ______________________________ **ULID:** ______________

**Directions:** Complete the attached problems, making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may use additional sheets of paper.

You may use books, notes, problem solutions that I've posted, *etc.* However, you may *not* use the help of others, and you may *not* help others. Asking questions on Internet forums is also *not* allowed. Please sign in the Academic Honestly section of this page to certify that you have followed these rules. Exams without a signature will be not be accepted.

**Submission:** To submit this exam, before 11:59pm on Tuesday, May 5, 2020, send an email:

- to `joshua.vaughan@louisiana.edu`
- with subject line `ULID-MCHE485-Final` where the `ULID` is your ULID and
- with your solutions attached as a *single* PDF file with filename `ULID-MCHE485-Final.pdf`, where `ULID` is your ULID.

Submissions with incorrect filenames or submitted as multiple images/PDFs will be rejected, and you will be asked to resubmit them.

Work submitted after 11:59pm will receive a penalty at the rate of 10 points per half-hour.

**Academic Honesty:**

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

By signing below, I certify that I have neither given nor received help on this exam:

______________________________________

**______ Problem 1 – 30 Points**

The system in Figure 1 is a rigid bar of mass $m$ and length $l$. It is connected to ground via a spring, $k$, and viscous damper, $c$, at a distance $a$ from the perfect pin about which it rotates. The spring is in equilibrium when $\theta = 0$. There is a pure torque, $\tau$, acting on the bar. Gravity also acts on the system.

a. Write the equations of motion for this system.

b. Write the *linearized* equations of motion for this system.

c. What is the natural frequency?

d. What is the damping ratio?

e. Assuming $\tau(t) = 0$, write the response, $\theta(t)$, to initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$.

f. Plot the response, $\theta(t)$, to the same initial conditions for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

g. Now, assume a harmonic input in $\tau(t)$. Write the transfer function from the amplitude of the input to the amplitude of the output.

h. Again assuming that $\tau(t)$ is a pure harmonic input, sketch the approximate frequency response for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses. Also indicate:

   i. Magnitude and phase as $\omega$ approaches 0.

   ii. Magnitude and phase as $\omega$ approaches infinity.

   iii. Magnitude when $\omega$ equals the natural frequency of the system, $\omega_n$.

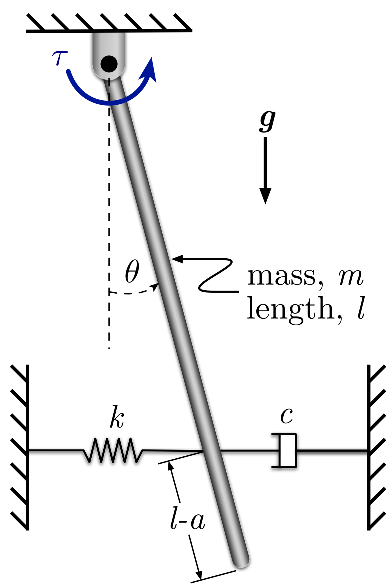


**Figure 1: A Pendulum with Spring and Damper**

**______ Problem 2 – 20 Points**

The system in Figure 2 consists of mass, $m$, connected to ground through a two springs, both of spring constant $k$, and a damper, $c$. A force, $f(t)$, acts on the system.

a. Write the equations of motion for this system.

b. What is the natural frequency?

c. What is the damping ratio?

d. Assuming $f(t) = 0$, sketch the response to initial conditions $x(0) = x_0$ and $\dot{x}(0) = -\dot{x}_0$ for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

e. Assuming $f(t) = \bar{f}\sin(\omega t)$, write the transfer function from the amplitude of the force, $\bar{f}$, to the amplitude of the response.

f. The forcing function was determined to *not* be a pure harmonic. However, a Fourier Analysis revealed that it could be adequately approximated as a linear combination of two pure harmonics, such that:

$$f(t) \approx \bar{f}_1 \sin(\omega_1 t) + \bar{f}_2 \sin(\omega_2 t)$$

where neither of the input frequencies match the natural frequency of the system ($\omega_1 \neq \omega_2 \neq \omega_n$). Write the time response, $x(t)$, to this approximation of the input.

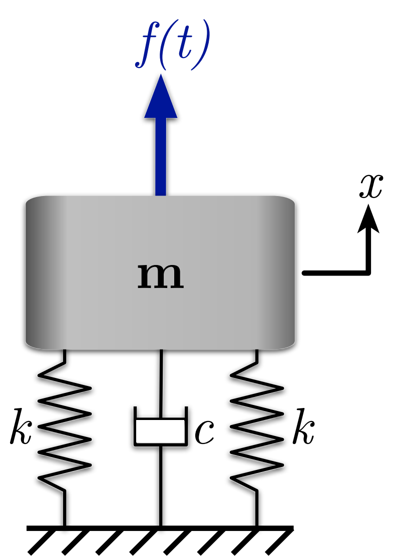


**Figure 2: A Forced Mass-spring-damper System**

______ **Problem 3 – 30 Points**

The model in Figure 3 is a model for a single-link robotic arm interacting with an object. The arm is modeled modeled as link of length $l$ with mass $m_1$. Its angle is represented by $\theta$. Pure torque $\tau$ drives the rotation. The object being manipulated is modeled as a mass, $m_2$, constrained to move on a frictionless surface. Its interaction with the arm is modeled via a spring, $k$, and a damper, $c$, attached between the two.

a. Write the equations of motion for this model in matrix form.

b. Assuming there is no damping ($c = 0$), set up and explain the solution procedure to find the natural frequencies and mode shapes. You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

c. Do you expect there to be a rigid-body mode? Why or why not?

d. For a given set of parameters, the natural frequencies and mode shapes were found to be:

$$\omega_1 = 0.00 \text{ rad/s} \quad \text{and} \quad X_1 = [5.00 \quad 0.50]^T$$
$$\omega_2 = 6.28 \text{ rad/s} \quad \text{and} \quad X_2 = [7.07 \quad -7.07]^T$$

Assuming that these are correct, plot (on separate sets of properly labeled axes) the approximate time responses, $\theta(t)$ and $x(t)$, to the initial conditions:

   i. $\theta(0) = \theta_0$, $x(0) = 0$, and all initial velocities are zero.

   ii. $\theta(0) = x(0) = 0$, $\dot{\theta}(0) = \dot{\theta}_0$, and $\dot{x}(0) = -0.1\dot{\theta} + \epsilon$, where $\epsilon$ is a small positive number.

e. Now, $c \neq 0$. Will the mode shapes change? Why or why not?

f. Set up and explain the solution procedure to find the natural frequencies and mode shapes for the damped case ($c \neq 0$). You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

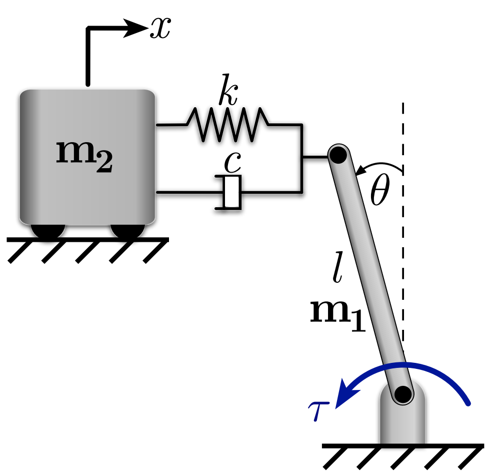


**Figure 3: A Simple Robotic Arm Model**

**______ Problem 4 – 20 Points**

The Ragin' Cajun Racing team recently got a set of lightweight, high-performance wheels for their car. The complete tire-wheel assembly has mass $m_1$. Unfortunately, someone forgot to balance the new wheels. The imbalance can be approximated by a second mass, $m_2$, offset from center by eccentricity, $e$. The wheel is rotating at angular velocity of $\omega$. The tire acts as a parallel spring, $k$, and damper, $c$, between the wheel and the road, where the displacement of that equivalent spring and damper can be approximated by the vertical motion of the wheel, $x$.

a. Write the equations of motion describing the vertical motion of the wheel center, as described by $x$.

b. Write the time response describing the vertical motion of the wheel, $x(t)$.

c. Assuming the system is underdamped, sketch the approximate frequency response. for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses. Also include:

   i. Magnitude as $\omega$ approaches 0.

   ii. Magnitude as $\omega$ approaches infinity.

   iii. Magnitude when $\omega$ equals the natural frequency of the system, $\omega_n$.

d. Write an expression describing the forces transmitted to the road. *Hint:* This can (and should) be done in terms of the time response, $x(t)$, you found above.

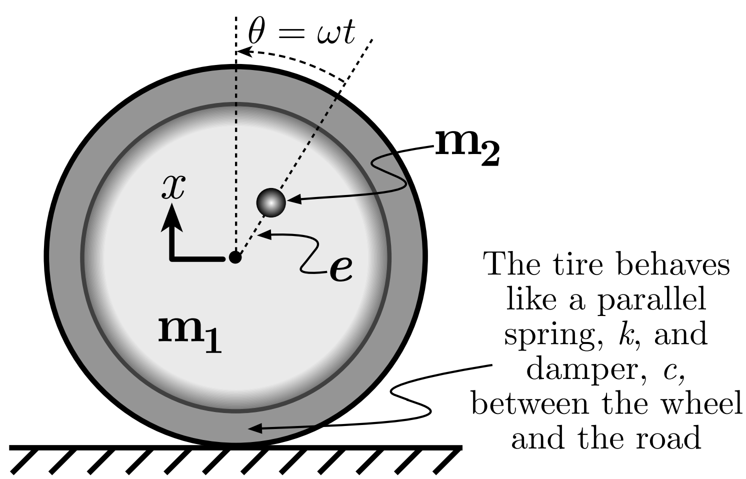


**Figure 4: An Unbalanced Formula SAE Wheel**

# Possibly Useful Equations

$$\bar{f} = m\bar{a}$$

$$I_0\bar{\alpha} = \sum \bar{M}_0$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

$$x(t) = ae^{i\omega_n t} + be^{-i\omega_n t}$$

$$x(t) = a\cos\omega_n t + b\sin\omega_n t$$

$$x(t) = e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\int u\ dv = uv - \int v\ du$$

$$\delta_{oc}V = \forall \sum$$

$$x(t) = \frac{\omega_n^2 \bar{y}}{\omega_n^2 - \omega^2}\sin(\omega t)$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) + \frac{\delta RD}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = Q_i, \quad i = 1, \ldots, n$$

$$M\ddot{X} + C\dot{X} + KX = F$$

$$\det\left(K - \omega^2 M\right) = 0 \qquad \left[K - \omega^2 M\right]\bar{X} = 0$$

$$2 + 2 = 2 \times 2 = 2^2 = 0\texttt{b}0100 = 0\texttt{x}2$$

$$i \equiv \sqrt{-1}$$

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau$$

$$x(t) = \int_0^t f(t-\tau)h(\tau)d\tau$$

$$f(t) = \sum_{n=0}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)$$

$$a_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)dt$$

$$a_0 = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)dt$$

$$V(\omega,\zeta) = e^{-\zeta\omega t_n}\sqrt{[C(\omega,\zeta)]^2 + [S(\omega,\zeta)]^2}$$

$$\tilde{X}_i = \frac{1}{\sqrt{X_i^T M X_i}}X_i$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$x(t) = c + e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$\sigma = \frac{1}{N}ln\left(\frac{x(0)}{x(Nt_p)}\right)$$

$$\sigma = ln\left(\frac{x(0)}{x(t_p)}\right)$$

$$\zeta = \frac{\sigma}{\sqrt{4\pi^2 + \sigma^2}}$$

$$\zeta = \frac{\sigma}{2\pi}$$

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}$$

$$\zeta \approx \frac{\delta_h}{2\omega_n}$$

$$E = mc^2$$

$$A = \begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix} \qquad B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}$$

$$\left[U^T M U\right]\ddot{H} + \left[U^T K U\right]H = U^T F\cos\omega t$$

# Possibly Useful Equations

$$
\begin{aligned}
x(t) &= -\frac{(2\zeta\omega\omega_n)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\cos\omega t + \frac{(\omega_n^2-\omega)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\sin\omega t \\
&= \left[\frac{\bar{f}}{m}\frac{1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\right]\cos(\omega t-\phi), \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\omega) &= \sqrt{\frac{\omega_n^4+(2\zeta\omega\omega_n)^2}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{i\phi}, \quad \text{where} \\
\phi &= \phi_1-\phi_2, \quad \text{where } \phi_1 = \tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right) \text{ and } \phi_2 = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\Omega) &= \sqrt{\frac{1+(2\zeta\Omega)^2}{(1-\Omega^2)^2+(2\zeta\Omega)^2}}e^{i\phi}, \quad \text{where} \\
\phi &= \tan^{-1}(2\zeta\Omega) - \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\omega) &= \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{-i\phi}, \quad \text{where} \\
\phi &= \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \frac{e\beta\omega^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\cos(\omega t-\phi), \\
&\quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \frac{e\beta\Omega^2}{\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}}\cos(\omega t-\phi), \quad \text{where} \\
\phi &= \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)
\end{aligned}
$$