MCHE 485: Mechanical Vibrations

Spring 2019 – Final Exam

Friday, May 10

Name: ______________________________ **ULID:** ______________

Directions: Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, which will be provided to you. No calculators are allowed.

Academic Honesty (just a reminder):

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

______ **Problem 1 – 30 Points**

The system in Figure 1 is a simple model of a vehicle suspension system. It consists of a mass, m, connected to a frictionless pin via a massless, rigid bar of length l. The tire is modeled as a connection to the road input, y, via a spring, k_t, and viscous damper, c_t. A spring, k, and damper, c, are connected between the bar at point A and ground at point B. The spring is in equilibrium when $\theta = 0$.

a. Write the equations of motion for this system. (*Hint:* Using the law of cosines, the distance between points A and B is defined by $|\bar{r}_{B/A}|^2 = a^2 + b^2 - 2ab\sin\theta$.)

b. Write the *linearized* equations of motion for this system.

c. What is the natural frequency?

d. What is the damping ratio?

e. Assume $y(t) = 0$ (*i.e.* it acts like another ground connection with respect to the mass). Write the response, $\theta(t)$, to initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = \dot{\theta}_0$.

f. Plot the response, $\theta(t)$, to the same initial conditions for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

g. Now, assume a harmonic input in $y(t)$. Write the transfer function from the amplitude of the input to the amplitude of the output.

h. Again assuming that $y(t)$ is a pure harmonic input, sketch the approximate frequency response for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses. Also indicate:

   i. Magnitude and phase as ω approaches 0.

   ii. Magnitude and phase as ω approaches infinity.

   iii. Magnitude when ω equals the natural frequency of the system, ω_n.

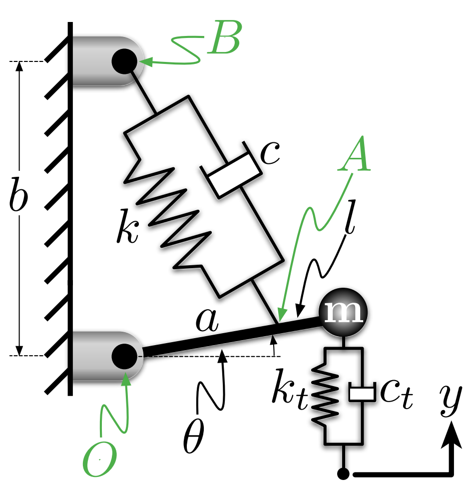

Figure 1: A Simple Vehicle Suspension Model

______ Problem 2 – 20 Points

The system in Figure 2 consists of mass, m, connected to ground through a massless, inflexible link of length l. There is a spring, k, and damper, c, attached between the mass and ground. A pure torque, $\tau(t)$, and gravity act on the system.

a. Write the equations of motion for this system.

b. What is the natural frequency?

c. What is the damping ratio?

d. Assuming $\tau(t) = 0$, sketch the response to initial conditions $\theta(0) = -\theta_0$ and $\dot{\theta}(0) = 0$ for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

e. Assuming $\tau(t) = \bar{\tau}e^{i\omega t}$, write the transfer function from the amplitude of the force, $\bar{\tau}$, to the amplitude of the response.

f. The forcing function was determined to *not* be a pure harmonic. However, a Fourier Analysis revealed that it could be adequately approximated as a linear combination of two pure harmonics, such that:

$$\tau(t) \approx \bar{\tau}_1 \sin(\omega_1 t) + \bar{\tau}_2 \sin(\omega_2 t)$$

where neither of the input frequencies match the natural frequency of the system ($\omega_1 \neq \omega_2 \neq \omega_n$). Write the time response, $\theta(t)$, to this approximation of the input.

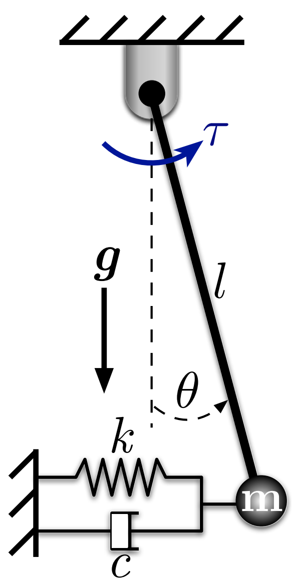

Figure 2: A Forced Pendulum

______ **Problem 3 – 30 Points**

Figure 3 is a model for a machining process. The cutting forces on the system are modeled via a spring, k, and a damper, c, attached between the endpoint and mass m_2. To approximate the deflection of the arm, it is divided into two links of identical length, l, and mass m, connected via a torsional spring, k_θ. The angles of the two links from horizontal are represented by θ_1 and θ_2, respectively. A pure torque, τ, drives the rotation.

a. Write the equations of motion for this model in matrix form.

b. Assuming there is no damping ($c = 0$), set up and explain the solution procedure to find the natural frequencies and mode shapes. You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

c. Do you expect there to be a rigid-body mode? Why or why not?

d. For a given set of parameters, the natural frequencies and mode shapes were found to be:

$$\omega_1 = 0.00 \text{ rad/s} \quad \text{and} \quad X_1 = [0.33 \;\; 0.33 \;\; 0.67]^T$$
$$\omega_2 = 2.67 \text{ rad/s} \quad \text{and} \quad X_2 = [0.32 \;\; 0.26 \;\; -0.75]^T$$
$$\omega_3 = 9.98 \text{ rad/s} \quad \text{and} \quad X_3 = [-0.89 \;\; 1.35 \;\; -0.02]^T$$

Assuming that these are correct, plot (on separate sets of properly labeled axes) the approximate time responses, $\theta_1(t)$, $\theta_2(t)$, and $x(t)$, to the initial conditions:

i. $\theta_1(0) = \theta_0$, $\theta_2(0) = \theta_0$, $x(0) = 2\theta_0$, and all initial velocities are zero.

ii. $\theta_1(0) = \theta_2(0) = x(0) = 0$, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = \dot{\theta}_0$, and $\dot{x}(0) = 2\dot{\theta}_0 + \epsilon$, where ϵ is a small positive number.

e. Now, $c \neq 0$. Will the mode shapes change? Why or why not?

f. Set up and explain the solution procedure to find the natural frequencies and mode shapes for the damped case ($c \neq 0$). You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

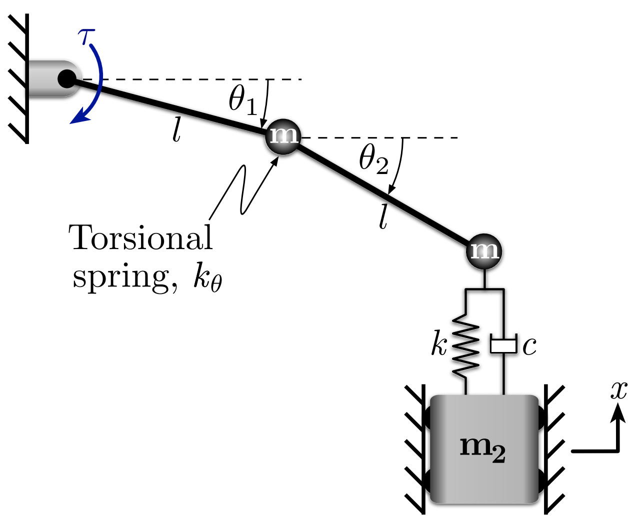

Figure 3: A Simple Machining Process Model

______ Problem 4 – 20 Points

The system in Figure 4 is a simple model of a wind turbine. It consists of a vertical beam with mass, m_1, connected to ground. The flexibility of the beam is approximately $k_{eq} = EA/l$, where l is its length. Its damping coefficient is approximately c_{eq}. The turbine blade is not perfectly balanced. Its imbalance can be approximated by a second mass, m_2, offset from center by eccentricity, e. The blade rotating at a constant angular velocity of ω.

a. Write the equations of motion describing the horizontal motion of the top of beam m_1, as described by x.

b. Assuming the system is underdamped, sketch the approximate frequency response. for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses. Also include:

   i. Magnitude as ω approaches 0.

   ii. Magnitude as ω approaches infinity.

   iii. Magnitude when ω equals the natural frequency of the system, ω_n.

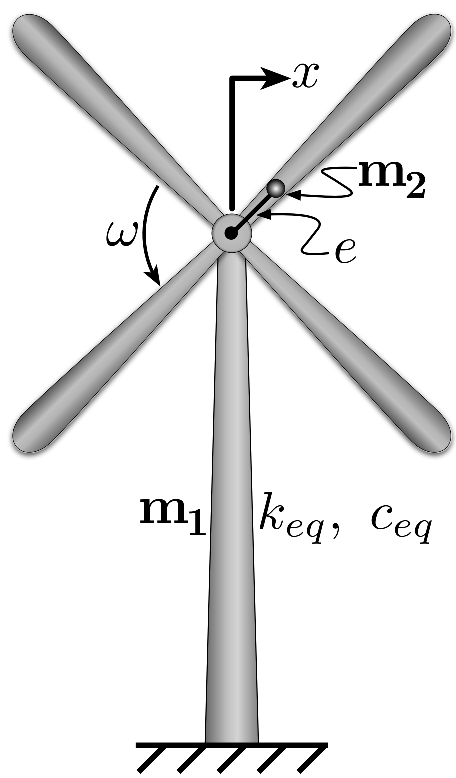

Figure 4: A Wind Turbine Model

Possibly Useful Equations

$$\bar{f} = m\bar{a}$$

$$I_0\bar{\alpha} = \sum \bar{M}_0$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

$$x(t) = ae^{i\omega_n t} + be^{-i\omega_n t}$$

$$x(t) = a\cos\omega_n t + b\sin\omega_n t$$

$$x(t) = e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\int u\ dv = uv - \int v\ du$$

$$\delta_{oc}V = \forall \sum$$

$$x(t) = \frac{\omega_n^2 \bar{y}}{\omega_n^2 - \omega^2}\sin(\omega t)$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) + \frac{\delta RD}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = Q_i, \quad i = 1, \ldots, n$$

$$M\ddot{X} + C\dot{X} + KX = F$$

$$\det\left(K - \omega^2 M\right) = 0 \qquad \left[K - \omega^2 M\right]\bar{X} = 0$$

$$2 + 2 = 2 \times 2 = 2^2 = 0\mathtt{b}0100 = 0\mathtt{x}2$$

$$i \equiv \sqrt{-1}$$

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau$$

$$x(t) = \int_0^t f(t-\tau)h(\tau)d\tau$$

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t) \\ a_n &= \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\cos(n\omega_0 t)dt \\ b_n &= \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)dt \\ a_0 &= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)dt \end{aligned}$$

$$V(\omega,\zeta) = e^{-\zeta\omega t_n}\sqrt{[C(\omega,\zeta)]^2 + [S(\omega,\zeta)]^2}$$

$$\tilde{X}_i = \frac{1}{\sqrt{X_i^T M X_i}}X_i$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$x(t) = c + e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$\sigma = \frac{1}{N}ln\left(\frac{x(0)}{x(Nt_p)}\right)$$

$$\sigma = ln\left(\frac{x(0)}{x(t_p)}\right)$$

$$\zeta = \frac{\sigma}{\sqrt{4\pi^2 + \sigma^2}}$$

$$\zeta = \frac{\sigma}{2\pi}$$

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}$$

$$\zeta \approx \frac{\delta_h}{2\omega_n}$$

$$E = mc^2$$

$$A = \begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix} \qquad B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}$$

$$\left[U^T M U\right]\ddot{H} + \left[U^T K U\right]H = U^T F\cos\omega t$$

Possibly Useful Equations

$$
\begin{aligned}
x(t) &= -\frac{(2\zeta\omega\omega_n)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\cos\omega t + \frac{(\omega_n^2-\omega)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\sin\omega t \\
&= \left[\frac{\bar{f}}{m}\frac{1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\right]\cos(\omega t-\phi), \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\omega) &= \sqrt{\frac{\omega_n^4+(2\zeta\omega\omega_n)^2}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{i\phi}, \quad \text{where} \\
\phi &= \phi_1-\phi_2, \quad \text{where } \phi_1 = \tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right) \text{ and } \phi_2 = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\Omega) &= \sqrt{\frac{1+(2\zeta\Omega)^2}{(1-\Omega^2)^2+(2\zeta\Omega)^2}}e^{i\phi}, \quad \text{where} \\
\phi &= \tan^{-1}(2\zeta\Omega) - \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\omega) &= \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{-i\phi}, \quad \text{where} \\
\phi &= \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \frac{e\beta\omega^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\cos(\omega t-\phi), \\
& \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \frac{e\beta\Omega^2}{\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}}\cos(\omega t-\phi), \quad \text{where} \\
\phi &= \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)
\end{aligned}
$$