MCHE 485: Mechanical Vibrations

Spring 2018 – Final Exam

Tuesday, May 1

Name: ______________________________ **CLID:** ____________

Directions: Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, which will be provided to you. No calculators are allowed.

Academic Honesty (just a reminder):

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

______ Problem 1 – 30 Points

The system in Figure 1 consists of a mass, m, connected to a frictionless pin via a massless, rigid bar of length l. It is also connected to an input, x, via viscous damper c. Gravity, g, is also acting on the system.

a. Write the equations of motion for this system.

b. Write the *linearized* equations of motion for this system.

c. What is the natural frequency?

d. What is the damping ratio?

e. Assume $x(t) = 0$ (*i.e.* it acts like another ground connection with respect to the mass). Write the response, $\theta(t)$, to initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$.

f. Plot the response, $\theta(t)$, to the same initial conditions for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

g. Now, assume a harmonic input in $x(t)$. Write the transfer function from the amplitude of the input to the amplitude of the output.

h. Assuming $x(t) = \bar{x}\cos\omega t$, write the time response, $\theta(t)$.

i. Again assuming that $x(t)$ is a pure harmonic input, sketch the approximate frequency response for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses. Also indicate:

   i. Magnitude and phase as ω approaches 0.

   ii. Magnitude and phase as ω approaches infinity.

   iii. Magnitude when ω equals the natural frequency of the system, ω_n.

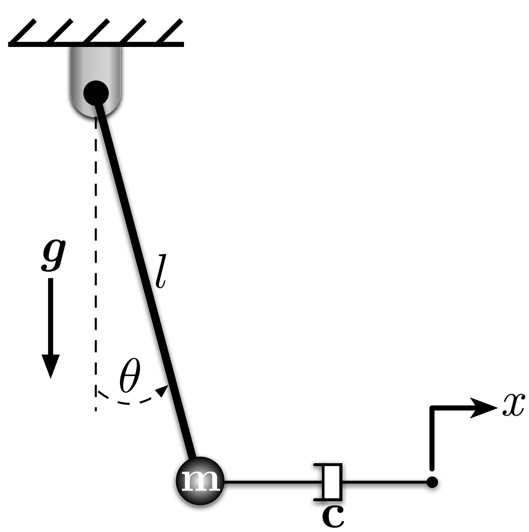

Figure 1: A Pendulum-like System with an Endpoint Input

______ Problem 2 – 30 Points

The system in Figure 2 consists of mass, m, connected to ground through a spring of spring constant k and a damper of damping coefficient c. The system rests on an frictionless incline of angle θ and force, $f(t)$, acts on the mass.

a. Write the equations of motion for this system. What is the natural frequency? What is the damping ratio?

b. Assuming $f(t) = 0$, sketch the response to initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

c. Assuming $f(t) = \bar{f}e^{i\omega t}$, write the transfer function from the amplitude of the force, $\bar{f}$, to the amplitude of the response.

d. The forcing function was determined to *not* be a pure harmonic. However, a Fourier Analysis revealed that it could be adequately approximated as a linear combination of two pure harmonics, such that:

$$f(t) \approx \bar{f}_1 \sin(\omega_1 t) + \bar{f}_2 \sin(\omega_2 t)$$

where neither of the input frequencies match the natural frequency of the system ($\omega_1 \neq \omega_2 \neq \omega_n$). Write the time response, $x(t)$, to this approximation of the input.

Now, the viscous damper is removed and the energy dissipation in the system modeled via friction with the incline with coefficient of friction, μ, as shown in Figure 3.

e. Write the equations of motion for this system.

f. Assuming $f(t) = 0$, sketch the response to initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Be sure to clearly label the axes, indicate any important features of the response, and to differentiate the response from the response with viscous damping that you sketched in part b. of this problem.

g. Does the angle of the incline affect how quickly the system response decays? Why or why not? If possible, support your answer via analysis of the equations of motion.

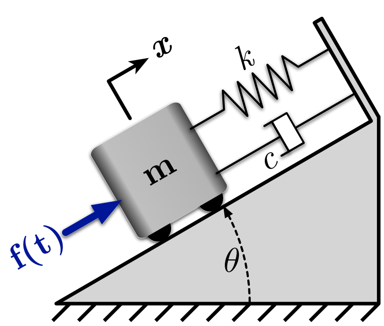

Figure 2: A Mass-Spring-Damper System on an Incline

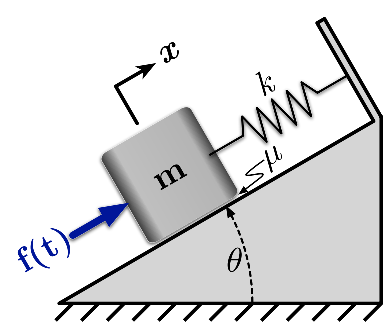

Figure 3: A Mass-Spring System on an Incline with Friction

______ Problem 3 – 30 Points

The model in Figure 4 could be used as a rough model for a robotic gripper. The fingers of the gripper are modeled as masses, m, attached to the end of a rigid, massless bars of length l. The angles of the two fingers are represented by θ_1 and θ_2 and both are attached to perfect, frictionless pins. Pure torques τ_1 and τ_2 at that pin joint control the gripper. The object being gripped is modeled as a mass, m_3, constrained to move on a frictionless surface. The interaction between the gripper fingers and the object is modeled via springs of spring constant k and dampers of damping coefficient c.

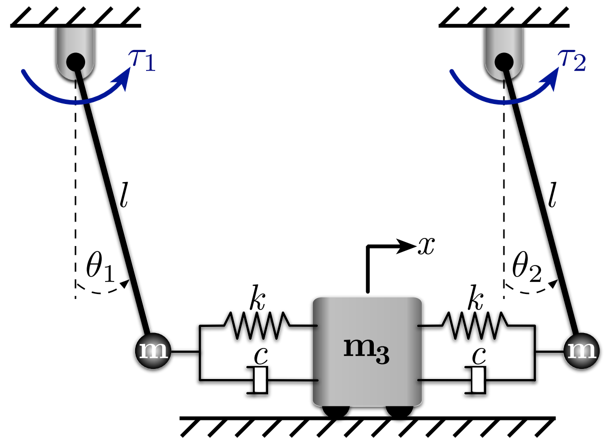

Figure 4: A Simple Robotic Gripper Model

a. Write the equations of motion for this model in matrix form.

b. Assuming there is no damping ($c = 0$), set up and explain the solution procedure to find the natural frequencies and mode shapes. You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

c. Do you expect there to be a rigid-body mode? Why or why not?

d. For a given set of parameters, the natural frequencies and mode shapes were found to be:

$$\begin{aligned} \omega_1 &= 0.00 \text{ rad/s} \quad \text{and} \quad X_1 = [5.00 \;\; 5.00 \quad 0.50]^T \\ \omega_2 &= 6.28 \text{ rad/s} \quad \text{and} \quad X_2 = [7.07 \;\; -7.07 \quad 0.00]^T \\ \omega_3 &= 8.89 \text{ rad/s} \quad \text{and} \quad X_3 = [5.00 \;\; 5.00 \;\; -0.50]^T \end{aligned}$$

Assuming that these are correct, plot (on separate sets of properly labeled axes) the approximate time responses, $\theta_1(t)$, $\theta_2(t)$, and $x(t)$, to the initial conditions:

i. $\theta_1(0) = \theta_0$, $\theta_2(0) = -\theta_0$, $x(0) = 0$, and all initial velocities are zero.

ii. $\theta_1(0) = \theta_2(0) = x(0) = 0$, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = \dot{\theta}_0$, and $\dot{x}(0) = -0.1\dot{\theta} + \epsilon$, where ϵ is a small positive number.

e. Now, $c \neq 0$. Will the mode shapes change? Why or why not?

f. Set up and explain the solution procedure to find the natural frequencies and mode shapes for the damped case ($c \neq 0$). You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

______ **Problem 4 – 10 Points**

The system in Figure 5 is a common model used for rotating machinery. It consists of a mass, m_1, connected to ground through a springs of spring constant k and a damper, c. The imbalance in the rotating portion of the machine can be approximated by a second mass, m_2, offset from center by eccentricity, e, and rotating at an angular velocity of ω.

a. Write the equations of motion describing the motion of m_1.

b. Assuming the system is underdamped, sketch the approximate frequency response. for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses. Also include:

   i. Magnitude as ω approaches 0.

   ii. Magnitude as ω approaches infinity.

   iii. Magnitude when ω equals the natural frequency of the system, ω_n.

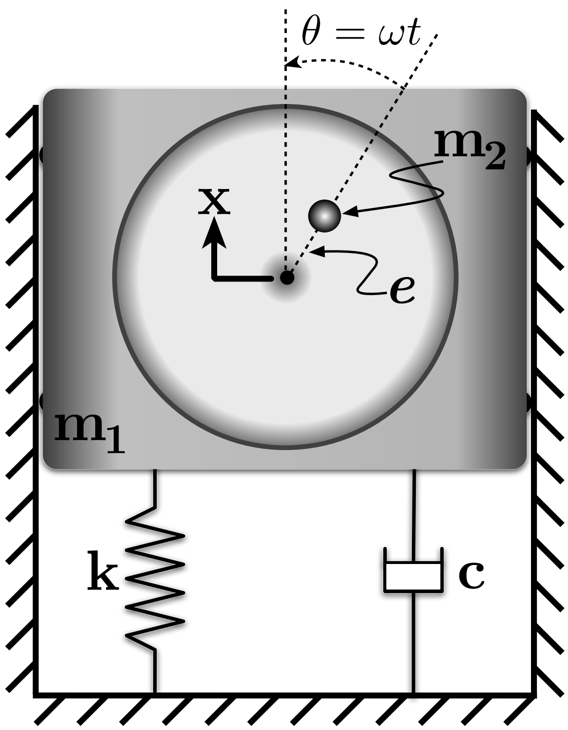

Figure 5: A Rotating Imbalance

Possibly Useful Equations

$$\bar{f} = m\bar{a}$$

$$I_0\bar{\alpha} = \sum \bar{M}_0$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

$$x(t) = ae^{i\omega_n t} + be^{-i\omega_n t}$$

$$x(t) = a\cos\omega_n t + b\sin\omega_n t$$

$$x(t) = e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\int u\ dv = uv - \int v\ du$$

$$\delta_{oc}V = \forall \sum$$

$$x(t) = \frac{\omega_n^2 \bar{y}}{\omega_n^2 - \omega^2}\sin(\omega t)$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) + \frac{\delta RD}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = Q_i, \quad i = 1, \ldots, n$$

$$M\ddot{X} + C\dot{X} + KX = F$$

$$\det\left(K - \omega^2 M\right) = 0 \qquad \left[K - \omega^2 M\right]\bar{X} = 0$$

$$2 + 2 = 2 \times 2 = 2^2 = 0\mathtt{b}0100 = 0\mathtt{x}2$$

$$i \equiv \sqrt{-1}$$

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau$$

$$x(t) = \int_0^t f(t-\tau)h(\tau)d\tau$$

$$f(t) = \sum_{n=0}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)$$

$$a_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)dt$$

$$a_0 = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)dt$$

$$V(\omega,\zeta) = e^{-\zeta\omega t_n}\sqrt{[C(\omega,\zeta)]^2 + [S(\omega,\zeta)]^2}$$

$$\tilde{X}_i = \frac{1}{\sqrt{X_i^T M X_i}}X_i$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$x(t) = c + e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$\sigma = \frac{1}{N}ln\left(\frac{x(0)}{x(Nt_p)}\right)$$

$$\sigma = ln\left(\frac{x(0)}{x(t_p)}\right)$$

$$\zeta = \frac{\sigma}{\sqrt{4\pi^2 + \sigma^2}}$$

$$\zeta = \frac{\sigma}{2\pi}$$

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}$$

$$\zeta \approx \frac{\delta_h}{2\omega_n}$$

$$E = mc^2$$

$$A = \begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix} \qquad B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}$$

$$\left[U^T M U\right]\ddot{H} + \left[U^T K U\right]H = U^T F\cos\omega t$$

Possibly Useful Equations

$$
\begin{aligned}
x(t) &= -\frac{(2\zeta\omega\omega_n)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\cos\omega t + \frac{(\omega_n^2-\omega)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\sin\omega t \\
&= \left[\frac{\bar{f}}{m}\frac{1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\right]\cos\left(\omega t-\phi\right), \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\omega) &= \sqrt{\frac{\omega_n^4+(2\zeta\omega\omega_n)^2}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{i\phi}, \quad \text{where} \\
\phi &= \phi_1-\phi_2, \quad \text{where } \phi_1 = \tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right) \text{ and } \phi_2 = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\Omega) &= \sqrt{\frac{1+(2\zeta\Omega)^2}{(1-\Omega^2)^2+(2\zeta\Omega)^2}}e^{i\phi}, \quad \text{where} \\
\phi &= \tan^{-1}\left(2\zeta\Omega\right) - \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
g(\omega) &= \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{-i\phi}, \quad \text{where} \\
\phi &= \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \frac{e\beta\omega^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\cos(\omega t-\phi), \\
&\quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \frac{e\beta\Omega^2}{\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}}\cos(\omega t-\phi), \quad \text{where} \\
\phi &= \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)
\end{aligned}
$$