MCHE 485: Mechanical Vibrations

Spring 2016 – Final Exam

Tuesday, May 3

Name: ____ **CLID:** ____________

Directions: Complete the attached problems making sure to clearly indicate your answer, show your work, and list any assumptions that you have made (with justification for them, if necessary). If you need extra space for any question, you may attach additional sheets of paper, which will be provided to you. No calculators are allowed.

Academic Honesty (just a reminder):

An essential rule in every class of the University is that all work for which a student will receive a grade or credit be entirely his or her own or be properly documented to indicate sources. When a student does not follow this rule, s/he is dishonest and s/he defeats the purpose of the course and undermines the goals of the University.

______ Problem 1 – 25 Points

The system in Figure 1 consists of a mass, m, connected to ground through through spring k and to an input, y, via damper c.

a. Write the equations of motion for this system.

b. What is the natural frequency?

c. Assume $y(t) = 0$ (*i.e.* it acts like another ground connection with respect to the mass). Write the response, $x(t)$, to initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$.

d. Plot the response, $x(t)$, to the same initial conditions for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes, indicate any important features of the responses, and differentiate between the responses.

e. Now, assume a harmonic input in $y(t)$. Write the transfer function from the amplitude of the input to the amplitude of the output.

f. Assuming $y(t) = \bar{y} \cos \omega t$, write the time response, $x(t)$.

g. Set up the solution procedure to determine the time response, $x(t)$, to the *velocity* input, $\dot{y}(t)$, shown in Figure 2. Define as much as possible based on the information you have been given. If there are integrals needed, you do not need to solve them, but do set the problem up such that it could be passed to a calculus student to do so. If terms in the integration will be zero, please be nice to the calculus student and indicate so.

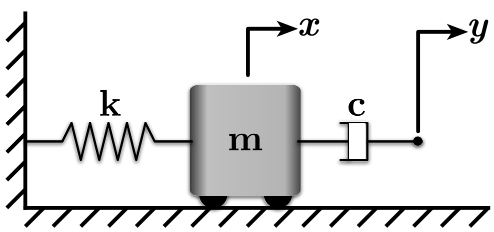

Figure 1: Mass-Spring-Damper System

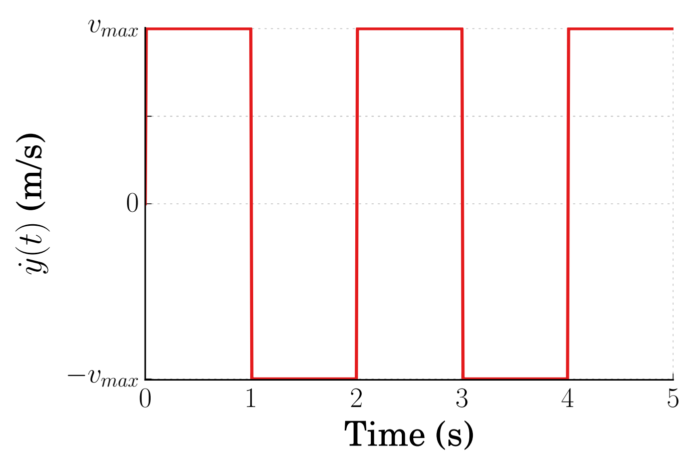

Figure 2: Velocity Command

______ Problem 2 – 20 Points

The system in Figure 3 consists of two masses, m_1 and m_2, affixed to opposite ends of a rigid, massless bar of length $l + a$. The distance from the frictionless pin to mass m_1 is l, and the distance from the pin to m_2 is a. Mass m_2 is also connected to ground via a spring, k, and a damper, c. There is an always horizontal force, $f(t)$, acting at m_1. You may ignore gravity.

a. Write the equations of motion describing this system.

b. Linearize the equations of motion about $\theta = 0$ and write them in state-space form.

c. Assume $f(t)$ is a pure harmonic input, $f(t) = \bar{f}e^{i\omega t}$. Sketch the approximate frequency response for damping ratios of $\zeta = 0.0$, $\zeta = 0.2$, and $\zeta = 0.7$. Be sure to clearly label the axes and differentiate between the responses. Also indicate:
   i. Magnitude and phase as ω approaches 0.
   ii. Magnitude and phase as ω approaches infinity.
   iii. Magnitude when ω equals the natural frequency of the system, ω_n.

A coworker suggests attaching the system in Figure 4 to the bar. He claims that this proposed sensor can measure its acceleration.

d. For the proposed sensor in Figure 4, write the transfer function between the sensor motion, $y(t)$, and the measurement, $x(t)$.

e. Explain how the sensor can be used to measure acceleration. (*Hint*: Using the transfer function from part c., write the relationship between $x(t)$ and $\ddot{y}(t)$.)

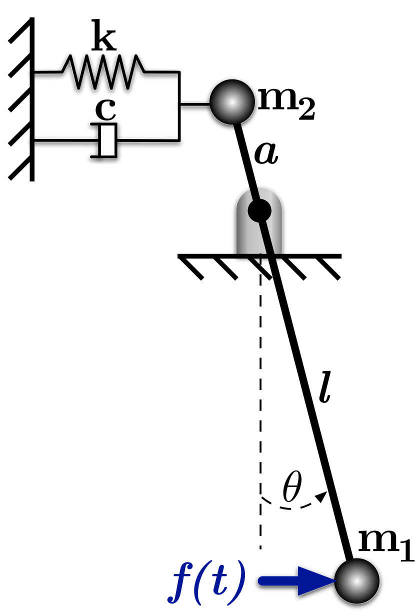

Figure 3: Forced, Pinned Barbell

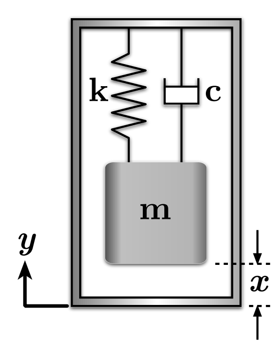

Figure 4: The Proposed Sensor

______ **Problem 3 – 15 Points**

The model sketched in Figure 5 contains a pulley with radius R and moment of inertia about its center of I_O. The pulley is connected to ground through a rotational spring, k, and rotational damper, c. A inextensible cable that passes over this pulley without slipping relative to the pulley connects masses m_1 and m_2.

a. Write the equations of motion describing this system.

b. Assume that the equilibrium angle of the rotational spring is $\theta = 0$ in absence of the two masses. Determine the equilibrium condition when the two-mass-cable system is attached to the pulley.

c. Assume that mass m_1 is displaced downward by x_0 from this equilibrium condition, then let go.

   i. Plot the response $\theta(t)$ to this initial displacement input.

   ii. Write the time response $\theta(t)$ to this initial displacement input.

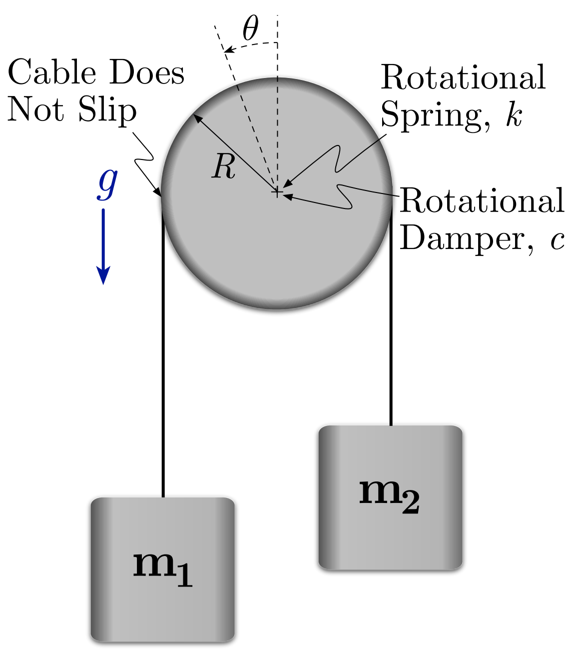

Figure 5: Two Masses Connected by a Wire Over a Pulley

______ **Problem 4 – 20 Points**

The model in Figure 6 could be used to simulate a flexible satellite with a thruster force, $f(t)$, acting on the center mass, m_2, which represents the main satellite body mass. The other two masses, m_1 and m_3 and the springs, k, and dampers c, are used to represent flexible appendages on the satellite, such at antennae or solar panels.

a. Write the equations of motion for this model in matrix form.

b. Assuming there is no damping ($c = 0$), set up and explain the solution procedure to find the natural frequencies and mode shapes. You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

c. For a given set of parameters, the natural frequencies and mode shapes were found to be:

$$\begin{aligned}
\omega_1 &= 0.00 \text{ rad/s} \quad \text{and} \quad X_1 = [\ 0.378 \quad 0.378 \quad 0.378]^T \\
\omega_2 &= 1.00 \text{ rad/s} \quad \text{and} \quad X_2 = [-0.707 \quad 0.000 \quad 0.707]^T \\
\omega_3 &= 1.18 \text{ rad/s} \quad \text{and} \quad X_3 = [-0.598 \quad 0.239 \quad -0.598]^T
\end{aligned}$$

Assuming that these are correct, plot (on separate sets of properly labeled axes) the approximate time responses, $x_1(t)$, $x_2(t)$, and $x_3(t)$, to the initial conditions:

i. $x_1(0) = x_2(0) = x_3(0) = x_0$ and all initial velocities are zero.

ii. $x_1(0) = x_2(0) = x_3(0) = 0$ and $\dot{x}_1(0) = v_0$, $\dot{x}_2(0) = 0$, and $\dot{x}_3(0) = -v_0$

iii. $x_1(0) = x_2(0) = x_3(0) = 0$ and $\dot{x}_1(0) = v_0$, $\dot{x}_2(0) = v_0 + \epsilon$, and $\dot{x}_3(0) = v_0$, where ϵ is a small positive number.

d. Now, $c \neq 0$. Will the mode shapes change? Why or why not?

e. Set up and explain the solution procedure to find the natural frequencies and mode shapes for the damped case ($c \neq 0$). You do not need to solve the complete problem, but all necessary information and steps must be clearly defined.

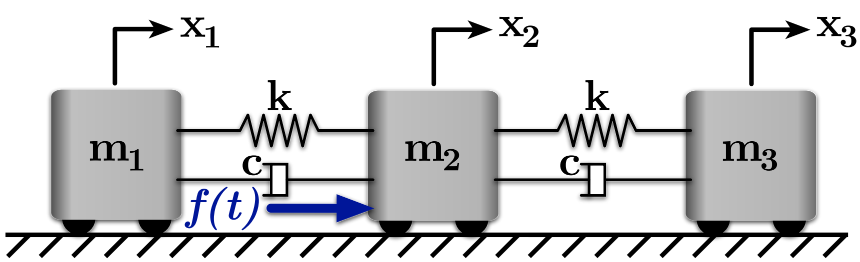

Figure 6: Flexible Satellite Model

______ Problem 5 – 20 Points

The system in Figure 7 contains a mass, m_t connected to ground by a spring, k. A massless and inflexible linkage of length l suspends a mass, m, from a perfect pin joint. There is a always-horizontal force, $f(t)$, acting on mass m.

a. Write the equations of motion for this system.

b. Write the linearized equations of motion for this system.

c. Set up and outline the solution procedure to determine the response to the force, $f(t)$. Include all information necessary up to the point of the actual algebra needed solve the problem. Be sure to fully define everything needed for the solution and fully explain the steps necessary.

d. For a given set of parameters, the natural frequencies and mode shapes for this system were found to be:

$$\omega_1 = 0.90 \text{ rad/s} \quad \text{and} \quad X_1 = [1.00 \quad 0.10]^T$$
$$\omega_2 = 2.47 \text{ rad/s} \quad \text{and} \quad X_2 = [1.00 \quad -2.59]^T$$

Can the force, $f(t)$, ever excite just a single mode? Why or why not?

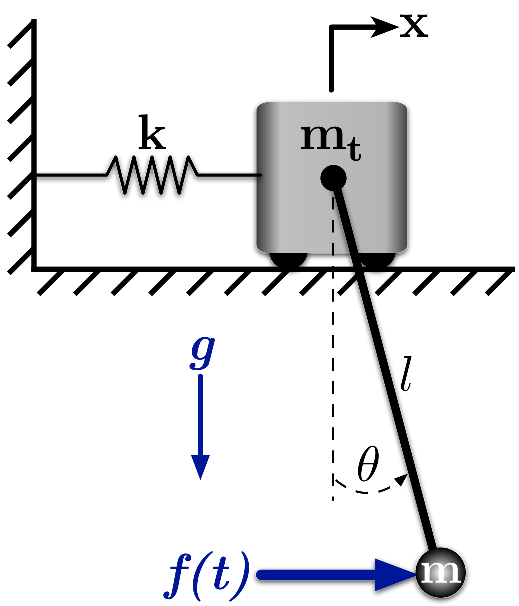

Figure 7: Forced Mass-Spring-Pendulum System

Possibly Useful Equations

$$\bar{f} = m\bar{a}$$

$$I_0\bar{\alpha} = \sum \bar{M}_0$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

$$x(t) = ae^{i\omega_n t} + be^{-i\omega_n t}$$

$$x(t) = a\cos\omega_n t + b\sin\omega_n t$$

$$x(t) = e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\int u\ dv = uv - \int v\ du$$

$$\delta_{oc}V = \forall \sum$$

$$x(t) = \frac{\omega_n^2\bar{y}}{\omega_n^2 - \omega^2}\sin(\omega t)$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) + \frac{\delta RD}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = Q_i, \quad i = 1, \ldots, n$$

$$M\ddot{X} + C\dot{X} + KX = F$$

$$\det\left(K - \omega^2 M\right) = 0 \qquad \left[K - \omega^2 M\right]\bar{X} = 0$$

$$2 + 2 = 2 \times 2 = 2^2 = 0\mathtt{b}0100 = 0\mathtt{x}2$$

$$i \equiv \sqrt{-1}$$

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau$$

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$$f(t) = \sum_{n=0}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)$$

$$a_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)dt$$

$$a_0 = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)dt$$

$$V(\omega,\zeta) = e^{-\zeta\omega t_n}\sqrt{[C(\omega,\zeta)]^2 + [S(\omega,\zeta)]^2}$$

$$\tilde{X}_i = \frac{1}{\sqrt{X_i^T M X_i}}X_i$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$x(t) = c + e^{-\zeta\omega_n t}\left[a\cos(\omega_d t) + b\sin(\omega_d t)\right]$$

$$\sigma = \frac{1}{N}ln\left(\frac{x(0)}{x(Nt_p)}\right)$$

$$\sigma = ln\left(\frac{x(0)}{x(t_p)}\right)$$

$$\zeta = \frac{\sigma}{\sqrt{4\pi^2 + \sigma^2}}$$

$$\zeta = \frac{\sigma}{2\pi}$$

$$\omega_p = \omega_n\sqrt{1 - 2\zeta^2}$$

$$\zeta \approx \frac{\delta_h}{2\omega_n}$$

$$E = mc^2$$

$$A = \begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix} \qquad B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}$$

$$\left[U^T M U\right]\ddot{H} + \left[U^T K U\right]H = U^T F\cos\omega t$$

Possibly Useful Equations

$$\begin{aligned} x(t) &= -\frac{(2\zeta\omega\omega_n)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\cos\omega t + \frac{(\omega_n^2-\omega)\frac{\bar{f}}{m}}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\sin\omega t \\ &= \left[\frac{\bar{f}}{m}\frac{1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\right]\cos(\omega t-\phi), \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right) \end{aligned}$$

$$\begin{aligned} g(\omega) &= \sqrt{\frac{\omega_n^4+(2\zeta\omega\omega_n)^2}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{i\phi}, \quad \text{where} \\ \phi &= \phi_1-\phi_2, \quad \text{where } \phi_1 = \tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right) \text{ and } \phi_2 = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right) \end{aligned}$$

$$\begin{aligned} g(\Omega) &= \sqrt{\frac{1+(2\zeta\Omega)^2}{(1-\Omega^2)^2+(2\zeta\Omega)^2}}e^{i\phi}, \quad \text{where} \\ \phi &= \tan^{-1}(2\zeta\Omega) - \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right) \end{aligned}$$

$$\begin{aligned} g(\omega) &= \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}e^{-i\phi}, \quad \text{where} \\ \phi &= \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right) \end{aligned}$$

$$\begin{aligned} x(t) &= \frac{e\beta\omega^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\cos(\omega t-\phi), \\ &\quad \text{where} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right) \end{aligned}$$

$$\begin{aligned} x(t) &= \frac{e\beta\Omega^2}{\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}}\cos(\omega t-\phi), \quad \text{where} \\ \phi &= \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right) \end{aligned}$$