Problem 3.1

3.1. Find the response of the system illustrated in Figure P3.1 to the input force shown.

Notice that this is identical to Example 3.1 from the book. The "actual" time scale has changed, but the relationship of the input to the period is the same.

We will still walk through setting up the Fourier Expansion of it. I'll include more detail in getting the response from this expansion.

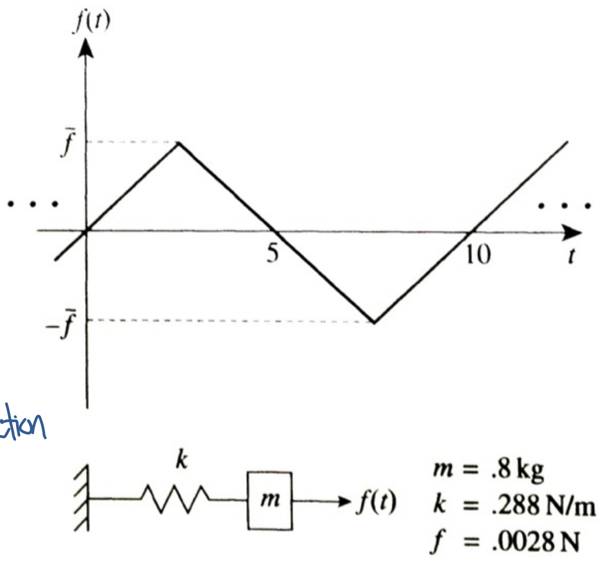


Figure P3.1

We know the transfer function of this system is:

$$g(\omega) = \frac{\bar f / m}{\omega_n^2 - \omega^2}$$

$$f(t) = \frac{\bar f - 0}{2.5 - 0} t + b = \frac{\bar f}{2.5} \quad 0 \le t \le 2.5 \left(0 \le t \le \frac{\tau_0}{4}, 0 \le t \le \frac{\pi}{2\omega_0}\right)$$

$$f(t) = \frac{-\bar f - \bar f}{7.5 - 2.5} t + b = \frac{-2\bar f}{5} t + b$$

$$f(5) = 0 = \frac{-2\bar f}{5}(5) + b \rightarrow b = 2\bar f$$

$$f(t) = -\frac{2}{5}\bar f t + 2\bar f, \quad 2.5 \le t \le 7.5 \left(\frac{\tau_0}{4} \le t \le \frac{3\tau_0}{4}, \frac{\pi}{2\omega_0} \le t \le \frac{3\pi}{2\omega_0}\right)$$

$$f(t) = \frac{0 + \bar f}{10 - 7.5} t + b$$

$$f(t) = \frac{\bar f}{2.5} t + b$$

$$f(10) = \frac{\bar f}{2.5}(10) + b = 0 \rightarrow b = -4\bar f$$

$$f(t) = \frac{2}{5}\bar f t - 4\bar f, \quad 7.5 \le t \le 10 \left(\frac{3\tau_0}{4} \le t \le \tau_0, \frac{3\pi}{2\omega_0} \le t \le \frac{2\pi}{\omega_0}\right)$$

$$f(t) = \begin{cases} \frac{2}{5}\bar f t & 0 \le t \le 2.5 \left(0 \le t \le \frac{\tau_0}{4}, 0 \le t \le \frac{\pi}{2\omega_0}\right) \\ -\frac{2}{5}\bar f t + 2\bar f & 2.5 \le t \le 7.5 \left(\frac{\tau_0}{4} \le t \le \frac{3\tau_0}{4}, \frac{\pi}{2\omega_0} \le t \le \frac{3\pi}{2\omega_0}\right) \\ \frac{2}{5}\bar f t - 4\bar f & 7.5 \le t \le 10 \left(\frac{3\tau_0}{4} \le t \le \tau_0, \frac{3\pi}{2\omega_0} \le t \le \frac{2\pi}{\omega_0}\right) \end{cases}$$

Odd function $(f(-x) = f(x))$ so only sine terms will be included in the solution. This means that all a_ns are 0.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \qquad \text{where} \quad b_n = \frac{\omega_0}{\pi} \int_0^{\frac{2\pi}{\omega_0}} f(t) \sin(n\omega_0 t)\, dt$$

Problem 3.1 (cont.)

$$f(t) = \begin{cases} \frac{2}{5}\bar{f}t & 0 \le t \le 2.5 \quad \left(0 \le t \le \frac{\tau_0}{4},\ 0 \le t \le \frac{\pi}{2\omega_0}\right) \longrightarrow \frac{2\omega_0 \bar{f}}{\pi} t \\ -\frac{2}{5}\bar{f}t + 2\bar{f} & 2.5 \le t \le 7.5 \quad \left(\frac{\tau_0}{4} \le t \le \frac{3\tau_0}{4},\ \frac{\pi}{2\omega_0} \le t \le \frac{3\pi}{2\omega_0}\right) \longrightarrow -\frac{2\omega_0}{\pi}\bar{f}t + 2\bar{f} \\ \frac{2}{5}\bar{f}t - 4\bar{f} & 7.5 \le t \le 10 \quad \left(\frac{3\tau_0}{4} \le t \le \tau_0,\ \frac{3\pi}{2\omega_0} \le t \le \frac{2\pi}{\omega_0}\right) \longrightarrow \frac{2\omega_0}{\pi}\bar{f}t - 4\bar{f} \end{cases}$$

$$b_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{\omega_0}{\pi}\left[\int_0^{\pi/2\omega_0} \frac{2\omega_0}{\pi}\bar{f}t\sin n\omega_0 t\,dt + \int_{\pi/2\omega_0}^{3\pi/2\omega_0}\left(\frac{-2\omega_0\bar{f}}{\pi}t + 2\bar{f}\right)\sin(n\omega_0 t)\,dt + \int_{3\pi/2\omega_0}^{2\pi/\omega_0}\left(\frac{2\omega_0}{\pi}\bar{f}t - 4\bar{f}\right)\sin(n\omega_0 t)\,dt\right]$$

1^{st} term $-$ $\int_0^{\pi/2\omega_0} \frac{2\omega_0}{\pi}\bar{f}t\sin n\omega_0 t\,dt$

$$= \frac{2\omega_0}{\pi}\bar{f}\int_0^{\pi/2\omega_0} \underbrace{t}_{u}\,\underbrace{\sin n\omega_0 t\,dt}_{dv}$$

$du = dt \qquad v = \frac{-1}{n\omega_0}\cos(n\omega_0 t)$

$$\frac{2\omega_0}{\pi}\bar{f}\left(-\frac{t\cos n\omega_0 t}{n\omega_0}\Big|_0^{\pi/2\omega_0} + \frac{1}{n\omega_0}\int_0^{\pi/2\omega_0}\cos n\omega_0 t\,dt\right)$$

$$\frac{2\omega_0}{\pi}\bar{f}\left(\frac{-\pi\cos\left(\frac{n\pi}{2}\right)}{2n\omega_0^2} + \frac{1}{(n\omega_0)^2}\sin(n\omega_0 t)\Big|_0^{\pi/2\omega_0}\right) = \frac{2\omega_0}{\pi}\left(\frac{\pi\cos\frac{n\pi}{2}}{2n\omega_0^2} + \frac{1}{(n\omega_0)^2}\left(\sin\frac{n\pi}{2}\right)\right)$$

$$= \frac{2\bar{f}}{n\pi\omega_0}\left(\frac{-\pi}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right)$$

2^{nd} term $-$ $\int_{\pi/2\omega_0}^{3\pi/2\omega_0} \underbrace{\left(\frac{-2\omega_0\bar{f}}{\pi}t + 2\bar{f}\right)}_{u}\underbrace{\sin(n\omega_0 t)\,dt}_{dv}$

$du = \frac{-2\omega_0}{\pi}\bar{f}\,dt \qquad v = \frac{-1}{n\omega_0}\cos(n\omega_0 t)$

$$\left(\frac{-2\omega_0}{\pi}\bar{f}t + 2\bar{f}\right)\left(\frac{-1}{n\omega_0}\cos(n\omega_0 t)\right)\Big|_{\pi/2\omega_0}^{\frac{3\pi}{2\omega_0}} + \frac{1}{n\omega_0}\int_{\pi/2\omega_0}^{3\pi/2\omega_0}\left(\frac{-2\omega_0}{\pi}\bar{f}\right)\cos(n\omega_0 t)\,dt$$

$$\left(-3\bar{f} + 2\bar{f}\right)\left(\frac{-1}{n\omega_0}\cos\frac{3n\pi}{2}\right) - \left(-\bar{f} + 2\bar{f}\right)\left(\frac{-1}{n\omega_0}\cos\frac{n\pi}{2}\right) + \left(\frac{-2\omega_0\bar{f}}{\pi(n\omega_0)^2}\sin(n\omega_0 t)\Big|_{\pi/2\omega_0}^{3\pi/2\omega_0}\right)$$

$$\frac{\bar{f}}{n\omega_0}\left(\cos\frac{3n\pi}{2} + \cos\frac{n\pi}{2}\right) - \frac{2\omega_0\bar{f}}{\pi(n\omega_0)^2}\left[\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

Problem 3.1 (cont.)

3rd term – $\left[\int_{3\pi/2\omega_0}^{2\pi/\omega_0} \underbrace{\left(\frac{2\omega_0}{\pi}\bar{f}t - 4\bar{f}\right)}_{u} \underbrace{\sin(n\omega_0 t)\,dt}_{dv}\right]$

$du = \frac{2\omega_0}{\pi}\bar{f}\,dt \qquad v = \frac{-1}{n\omega_0}\cos(n\omega_0 t)$

$$\left(\frac{2\omega_0}{\pi}\bar{f}t - 4\bar{f}\right)\left(\frac{-1}{n\omega_0}\cos(n\omega_0 t)\right)\Bigg|_{\frac{3\pi}{2\omega_0}}^{2\pi/\omega_0} + \frac{2\omega_0\bar{f}}{\pi(n\omega_0)}\int_{3\pi/2\omega_0}^{2\pi/\omega_0}\cos(n\omega_0 t)\,dt$$

$$\cancelto{0}{(4\bar{f}-4\bar{f})}(\_) - (3\bar{f}-4\bar{f})\left(\frac{-1}{n\omega_0}\cos\frac{3n\pi}{2}\right) + \frac{2\omega_0\bar{f}}{\pi(n\omega_0)^2}\left[\cancelto{0}{\sin(2n\pi)} - \sin\left(\frac{3n\pi}{2}\right)\right]$$

$$\frac{-\bar{f}}{n\omega_0}\cos\left(\frac{3n\pi}{2}\right) - \frac{2\omega_0\bar{f}}{\pi(n\omega_0)^2}\sin\left(\frac{3n\pi}{2}\right)$$

Total – $\frac{\omega_0}{\pi}\Big[\frac{2\bar{f}}{n\pi\omega_0}\left(\cancel{-\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right)} + \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right)$ ← 1st term

$+ \cancel{\frac{\bar{f}}{n\omega_0}\left(\cos\frac{3n\pi}{2} + \cos\frac{n\pi}{2}\right)} - \frac{2\omega_0\bar{f}}{\pi(n\omega_0)^2}\left[\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right]$ ← 2nd term

$\cancel{\frac{-\bar{f}}{n\omega_0}\cos}\left(\frac{3n\pi}{2}\right) - \frac{2\omega_0\bar{f}}{\pi(n\omega_0)^2}\sin\left(\frac{3n\pi}{2}\right)\Big]$ ← 3rd term

$$= \frac{\omega_0}{\pi}\left[-\frac{4\omega_0\bar{f}}{\pi(n\omega_0)^2}\sin\left(\frac{3n\pi}{2}\right) + \frac{4\bar{f}}{n^2\pi\omega_0}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{4\bar{f}}{(n\pi)^2}\sin\left(\frac{3n\pi}{2}\right) + \frac{4\bar{f}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{4\bar{f}}{(n\pi)^2}\underbrace{\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right]}$$

notice that this term is ± 2 for odd n and $=0$ for even n

Now, we can write $f(t) = \frac{4\bar{f}}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right)\sin(n\omega_0 t)$ ← We could also write this like on p.408 of the book

The transfer function of the system is $g(\omega) = \frac{1/m}{\omega_n^2 - \omega^2}$

So, we can write the response as:

$$\boxed{x(t) = \frac{4\bar{f}}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right)\left(\frac{\omega_n^2}{m(\omega_n^2 - (n\omega_0)^2)}\right)\sin(n\omega_0 t)\right]}$$

Forced Response via Convolution Integral (Sec 3.3)

The Fourier Analysis gives us a way to address periodic inputs, but...

Q: How can we determine the response to any input?
- Look at the response to a single impulse
- Show/know that we can approximate any input as a series of impulses
- Use superposition to sum the responses to this series of impulses

Q: What is an impulse?

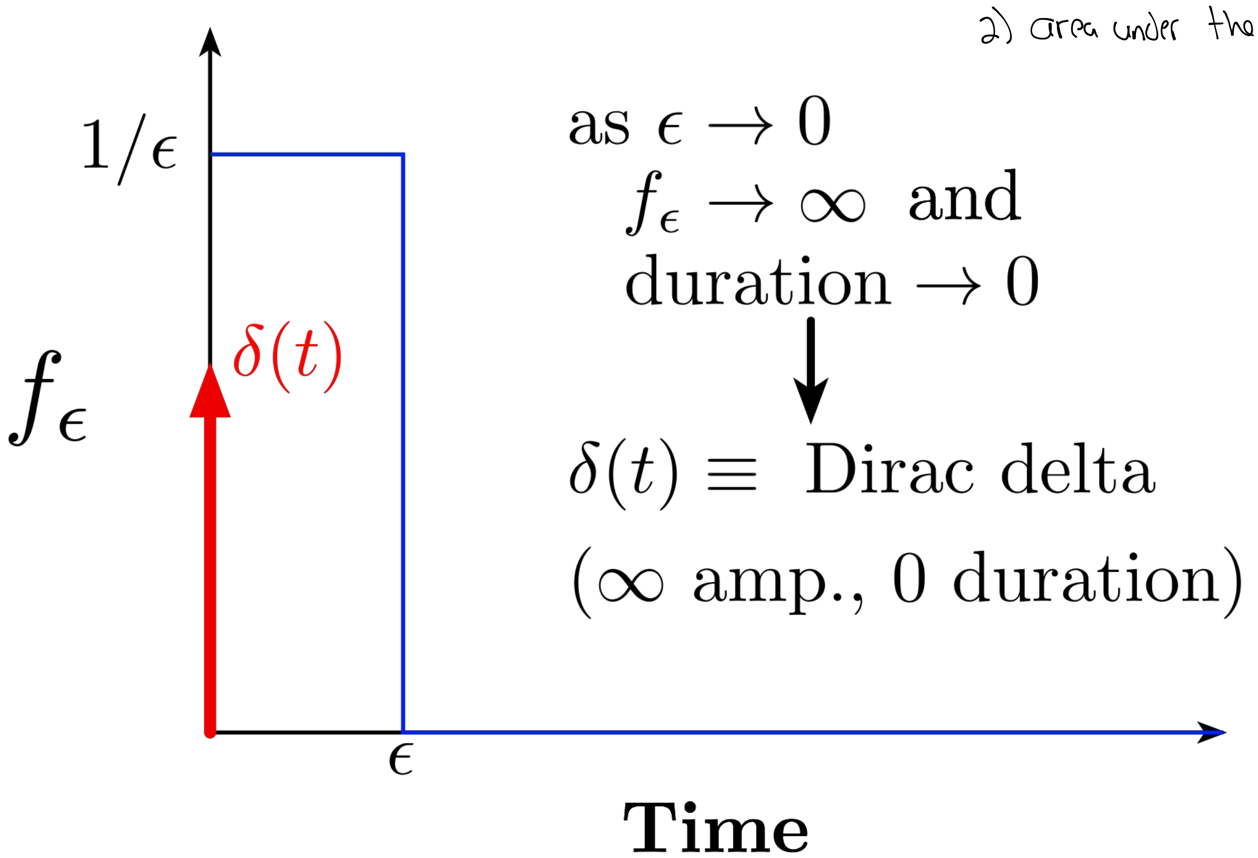
intuitively - a "high" magnitude input for a "short" time

strictly - an infinite magnitude input for zero time

An approximation

$$f_\epsilon = \begin{cases} 1/\epsilon , & 0 \le t \le \epsilon \\ 0 , & t > \epsilon \end{cases}$$

Notice
1) as ϵ gets smaller
   - amp increases
   - duration decreases
2) area under the curve is always 1



Q: What is the response to an impulse?

Impulse-momentum: $F = ma = m\frac{dv}{dt}$

$$F\,dt = m\,dv \longrightarrow \int F\,dt = \int m\,dv$$

$$\int F\,dt = mv\Big|_{t_0}^{t_1} = m(v_1 - v_0)$$

Forced Response via Convolution Integral (cont.)

Let's look at the response of a direct force system to our approx. impulse input

$$m\left(\dot{x}(\epsilon)-\dot{x}(0)\right)=\underbrace{\int_0^{\epsilon}\frac{1}{\epsilon}\,dt}$$

Let $\dot{x}(0)=0$

(really $\dot{x}(0^+)=0$ as $\epsilon\to 0$)

$$m\dot{x}(\epsilon)=1$$

as $\epsilon\to 0$ $\quad m\dot{x}(0^+)=1$

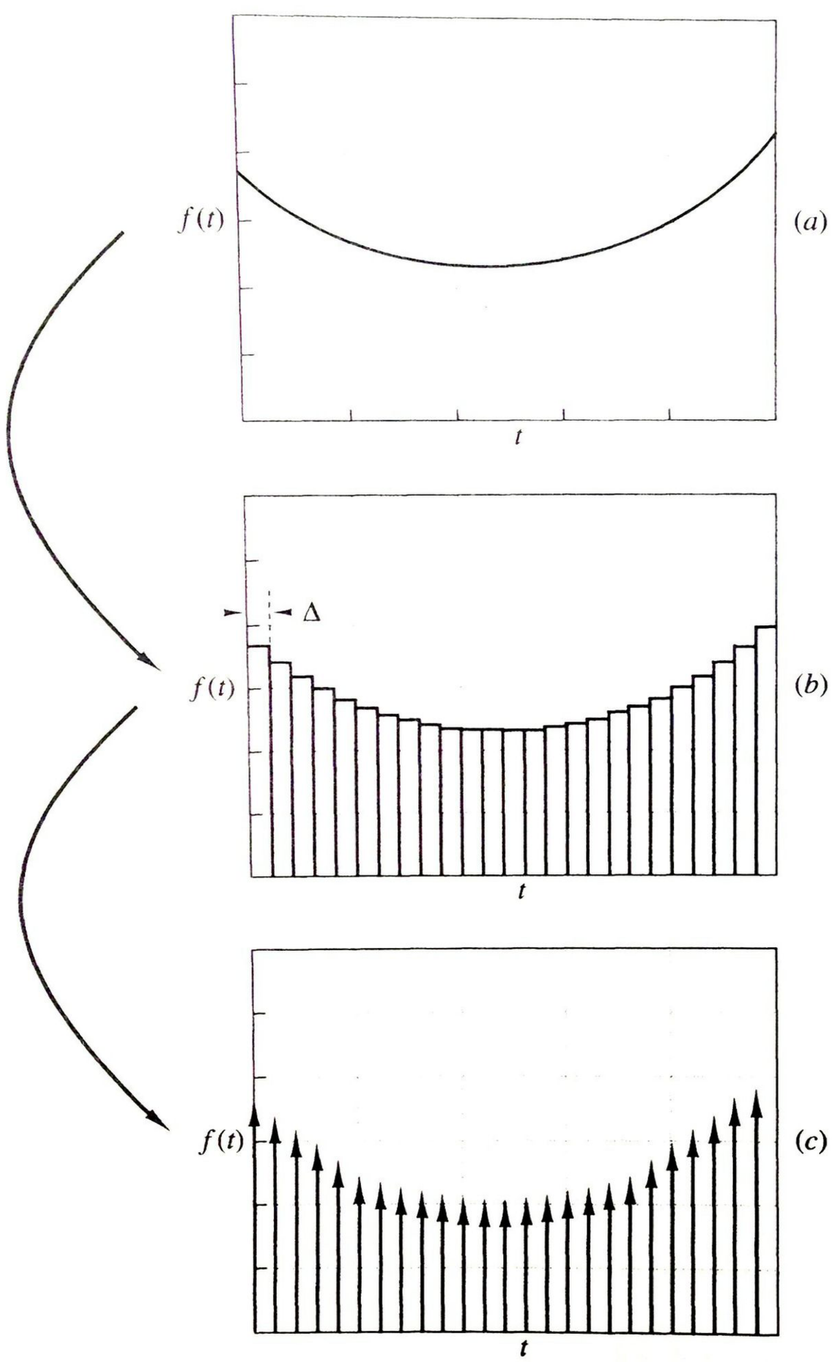
$$\dot{x}(0^+)=\frac{1}{m}$$

So, as $\epsilon\to 0$, the response becomes just a change in initial conditions

$$\left[\ x(0)=0 \ \text{ and } \ \dot{x}(0)=\frac{1}{m}\right]$$

So, the response is just free vibration ⟵ We know this!!

Q: How could we use this to solve for any input?



Take a given input

Divide it into segments of duration Δ

If Δ is small, the the top can be approx as flat and the area of a given segment is

$$\text{area}=\underbrace{f(n\Delta)\Delta}_{f(t)\,dt}$$

Now, replace these segments by a series of dirac delta functions

Scale each by the value of the function at that time location

1st impulse: $\left[f(0)\Delta\right]\delta(t)$

The response to the original function is just the sum of the responses to those impulses.

Forced Response via Convolution Integral (cont.)

Q: What is the response of a direct-force mass-spring-damper system to a dirac delta input?

$$\underbrace{m\ddot{x}+c\dot{x}+kx=\delta(t)}$$

This is equivalent to: $m\ddot{x}+c\dot{x}+kx=0$ with $x(0)=0$ and $\dot{x}(0)=\frac{1}{m}$

Q: What is the general solution to this ODE?

$$x(t)=e^{-\zeta\omega_n t}\left(a\cos\omega_d t+b\sin\omega_d t\right)$$ ← plug in initial conditions to find

$$\underbrace{=\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)}$$

This is the impulse response. $h(t)\equiv\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)$ ← Note: This will change based on the system

Now, let's look back at the original, general input:

$0\le t\le\Delta$ $\quad x(t)=\underbrace{[f(0)\Delta]}_{\text{scaling}}\underbrace{[h(t)]}_{\text{response}}=\frac{f(0)\Delta}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$

$\Delta\le t\le 2\Delta$ $\quad x(t)=\underbrace{f(0)\Delta h(t)}_{\text{response to 1st impulse}}+\underbrace{f(\Delta)\Delta h(t-\Delta)}_{\text{response to 2nd impulse}}=f(0)\Delta h(t)+\frac{f(\Delta)\Delta}{m\omega_d}e^{-\zeta\omega_n(t-\Delta)}\sin\omega_d(t-\Delta)$

impulse response is delayed by Δ to match the time the impulse occurs.

Now, let's look at the n^{th} interval $\left((n-1)\Delta\le t\le n\Delta\right)$

$$x(t)=\sum_{i=0}^{n-1}f(i\Delta)\Delta h(t-i\Delta)$$

Q: How can we write this as $\Delta\to 0$?

$\sum\to\int$ $\quad\Delta\to d\tau$ $\longrightarrow$ $x(t)=\int_0^t f(\tau)h(t-\tau)d\tau$ } Convolution Integral

For this system: $x(t)=\int_0^t f(\tau)\frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$

We can also write this as: $x(t)=\int_0^t f(t-\tau)\,h(\tau)d\tau$ (See book for derivation)

Key Point: Remember that we're just adding a series of impulse responses.