Chapter 3 - Non-Harmonic Excitations & Impact

So far, we've only looked at:

- Free responses (to initial conditions)
- Steady-state responses to harmonic (pure sine or cosine) inputs

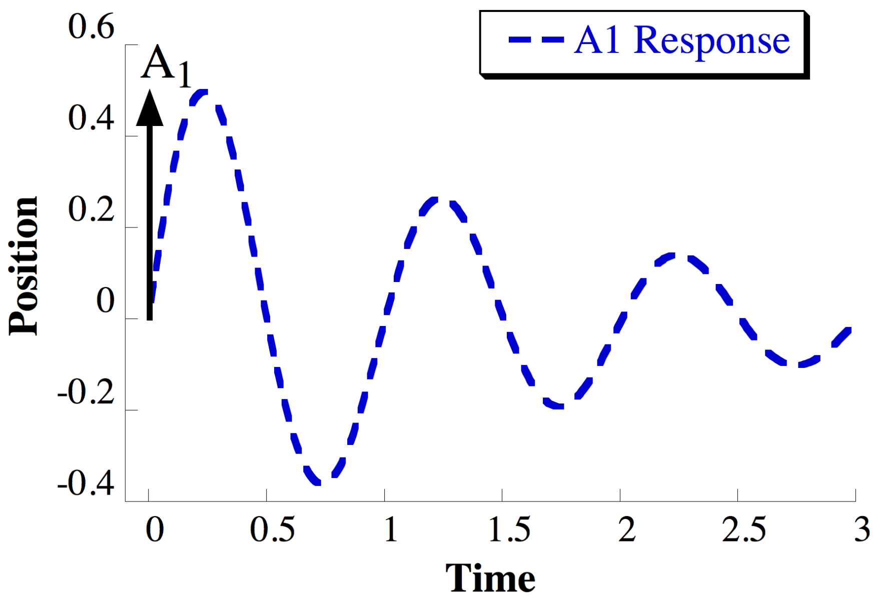
In reality, we often have non-harmonic inputs and/or care about transient response.

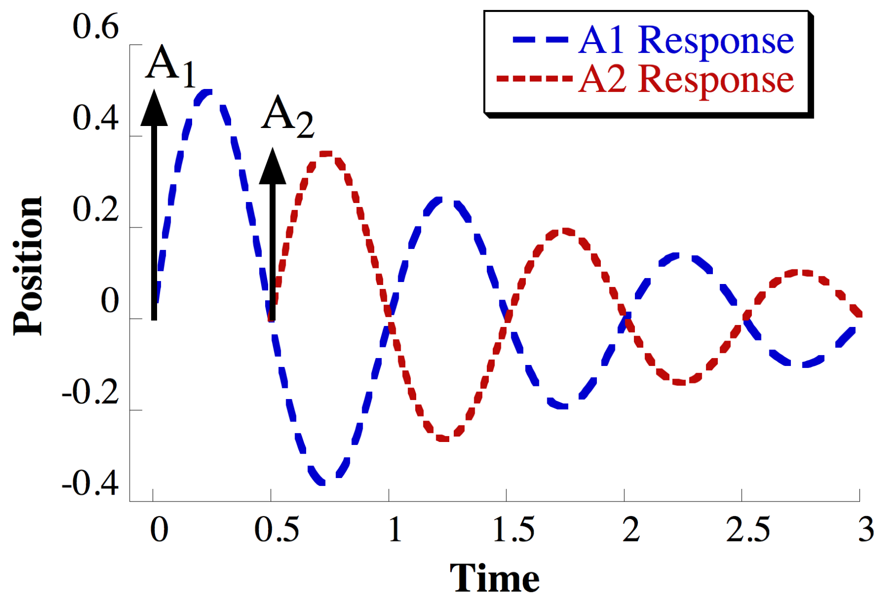
This chapter addresses those points. All of the techniques we'll learn are based around superposition.

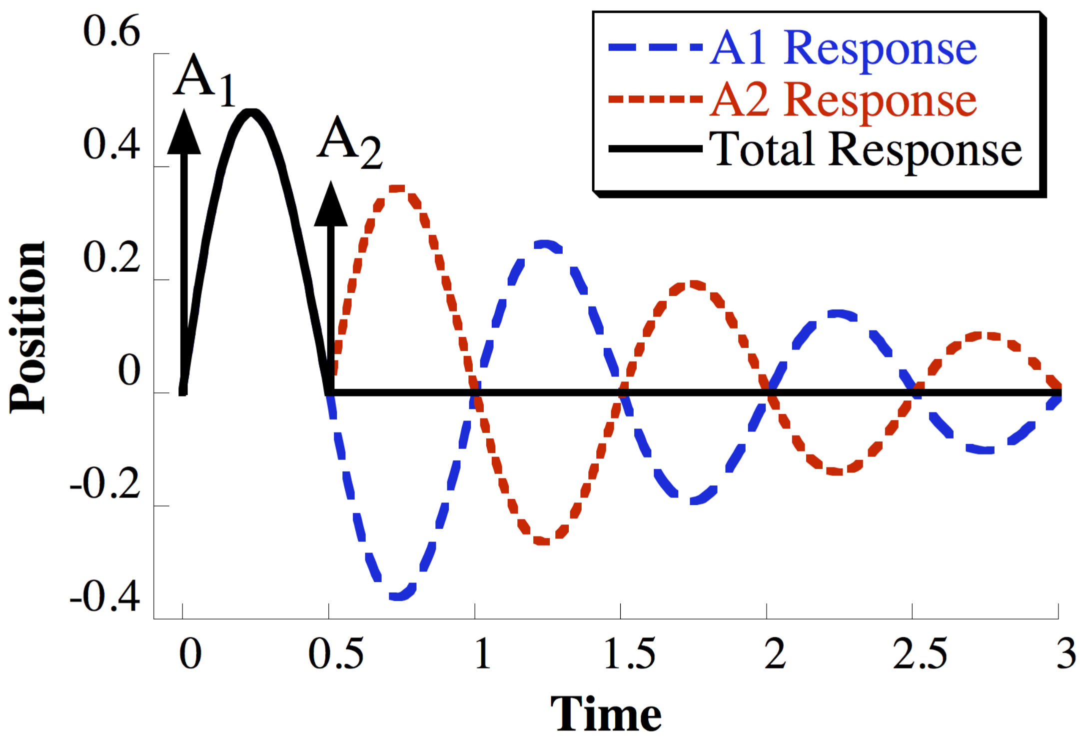
Q: What is superposition?

Most simply put:

If f(x1) = y1 and f(x2) = y2, then f(x1 + x2) = (y1 + y2)







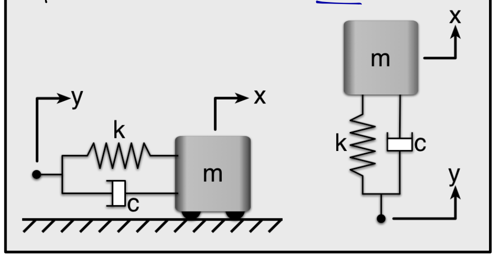
Fourier Series (Sec. 3.2)

We can use superposition to get information about a system's response to "non-pure" harmonic excitations.

Q: What's the response of an undamped, seismically excited system to:

$$y(t) = \frac{8}{\pi^2}\sin(\omega t) - \frac{32}{36\pi^2}\sin(3\omega t)$$

Aside: What does this mean?



We know that $\ddot{x} + \omega_n^2 x = \omega_n^2 y$

match $x(t)$ to the form of $y(t)$

$$x(t) = b_1 \sin(\omega t) + b_2 \sin(3\omega t)$$

As usual diff. and plug into the eq of motion

$$\dot{x}(t) = b_1 \omega \cos(\omega t) + 3 b_2 \omega \cos(3\omega t)$$

$$\ddot{x}(t) = -b_1 \omega^2 \sin(\omega t) - 9 b_2 \omega^2 \sin(3\omega t)$$

$$\left[-b_1\omega^2 \sin(\omega t) - 9 b_2 \omega^2 \sin(3\omega t)\right] + \omega_n^2\left[b_1 \sin(\omega t) + b_2 \sin(3\omega t)\right] = \omega_n^2\left[\frac{8}{\pi^2}\sin(\omega t) - \frac{32}{36\pi}\sin(3\omega t)\right]$$

Collect $\sin(\omega t)$ and $\sin(3\omega t)$ terms

$$b_1(\omega_n^2 - \omega^2)\sin(\omega t) + b_2(\omega_n^2 - 9\omega^2)\sin(3\omega t) = \omega_n^2\left[\frac{8}{\pi^2}\sin(\omega t) - \frac{32}{36\pi}\sin(3\omega t)\right]$$

$\sin(\omega t)$ and $\sin(3\omega t)$ are linearly independent, so we have to match coeff. to solve

$$b_1(\omega_n^2 - \omega^2) = \frac{8\omega_n^2}{\pi^2} \quad \text{and} \quad b_2(\omega_n^2 - 9\omega^2) = \frac{-32\omega_n^2}{36\pi^2}$$

We find $b_1 = \frac{8}{\pi^2}\frac{\omega_n^2}{\omega_n^2 - \omega^2}$ and $b_2 = \frac{-32}{36\pi}\frac{\omega_n^2}{\omega_n^2 - 9\omega^2}$ } plug back into $x(t) = b_1 \sin(\omega t) + b_2 \sin(3\omega t)$

This is the exact response we would get if we solve:

$$\ddot{x} + \omega_n^2 x = \frac{8}{\pi^2}\sin(\omega t) \quad \text{and} \quad \ddot{x} + \omega_n^2 x = \frac{-32}{36\pi^2}\sin(3\omega t)$$

then combine them

Q: Can we use this fact to represent an arbitrary input?

Fourier Series (cont.)

Mr. Fourier showed that we can represent and well-behaved periodic signal as a infinite sum of sines and cosines.

$f(t) = f(t+T)$

where T is the function's period

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$\omega_0 =$ lowest frequency in $f(t)$

Q: What are challenges to implementing this?

Need an infinite number of terms.

In practice, we can get good approximations (and sometimes exact solutions) with far fewer.

Q: What should a_n and b_n be?

(see the derivation in the book on p156-159)

$$a_0 = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} f(t)\,dt$$

This term will only be nonzero if there is a DC offset.

$$a_n = \frac{\omega_0}{\pi} \int_0^{2\pi/\omega_0} f(t)\cos(n\omega_0 t)\,dt$$

This is just the period of the function

We only need to look at the contributions from a single cycle, because the function is periodic.

$$b_n = \frac{\omega_0}{\pi} \int_0^{2\pi/\omega_0} f(t)\sin(n\omega_0 t)\,dt$$

Fourier Analysis in One Figure

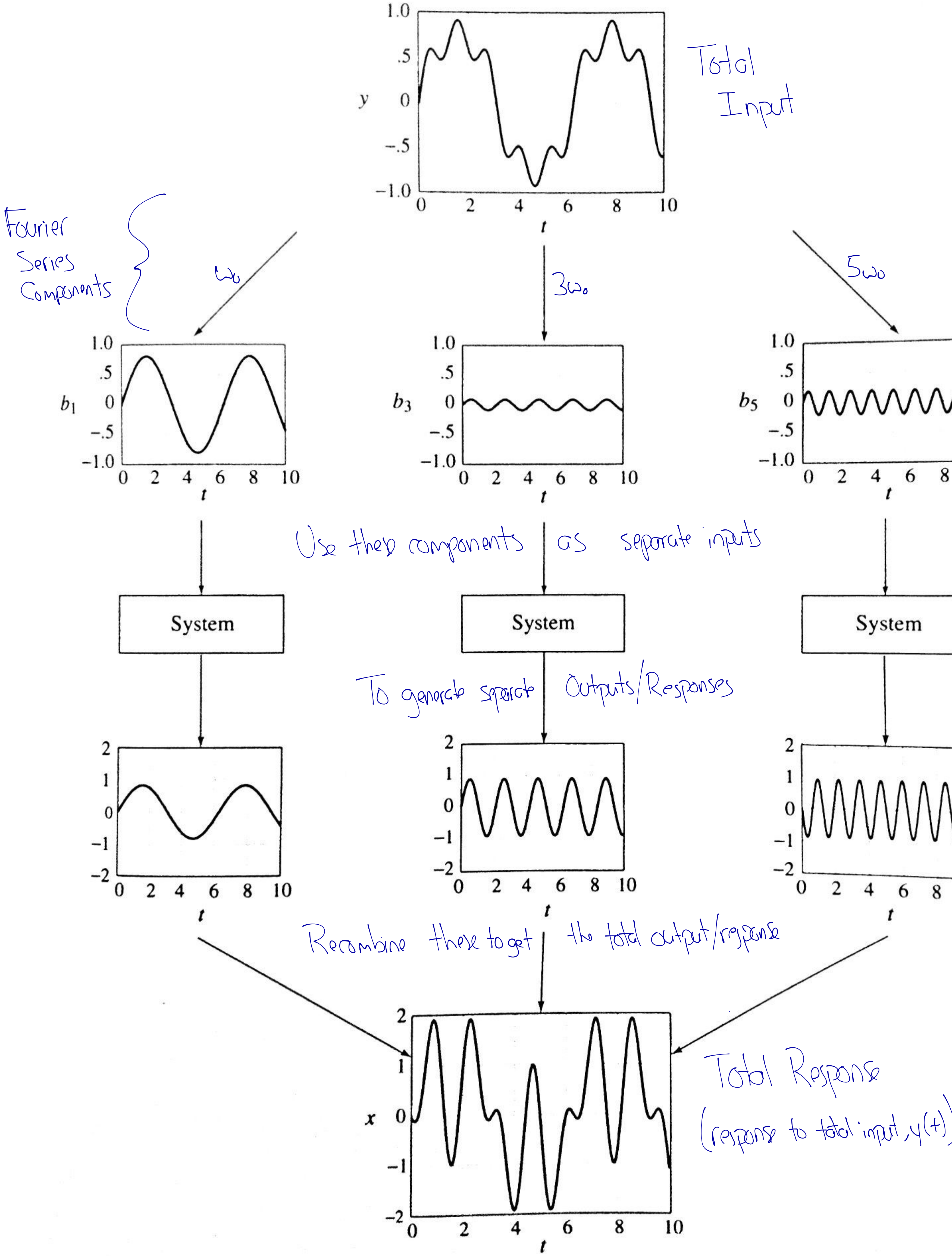


Figure 3.3 Schematic of how Fourier analysis works