Problem 1.100

1.100. Find the equation of motion for the system illustrated in Figure P1.100.

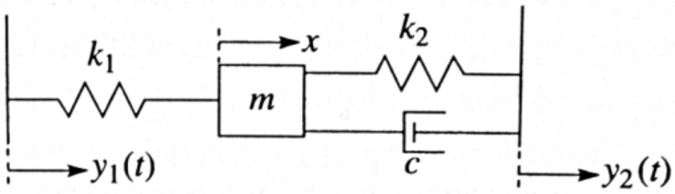
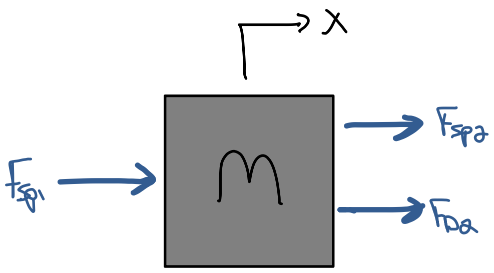


Figure P1.100

FBD



$F_{sp1} = k_1\delta_1$, δ_1 is the difference between y_1 and x

$= k_1(y_1 - x)$

$F_{sp2} = k_2\delta_2$, δ_2 is the difference between y_2 and x

$= k_2(y_2 - x)$

$F_{D2} = c\dot{\delta}_2 = c(\dot{y}_2 - \dot{x})$

So,

$$m\ddot{x} = k_1(y_1 - x) + k_2(y_2 - x) + c(\dot{y}_2 - \dot{x})$$

$$\boxed{m\ddot{x} + c\dot{x} + (k_1 + k_2)x = k_1 y_1 + k_2 y_2 + c_2\dot{y}_2}$$

Problem 2.31

2.31. Find the equations of motion and natural frequency for system shown in Figure P2.31 (neglect gravity). What would f need to be for there to be no overall excitation of the system? ($\bar{f}$: newtons, $\bar{y}$: meters). The system is freely hinged at O.

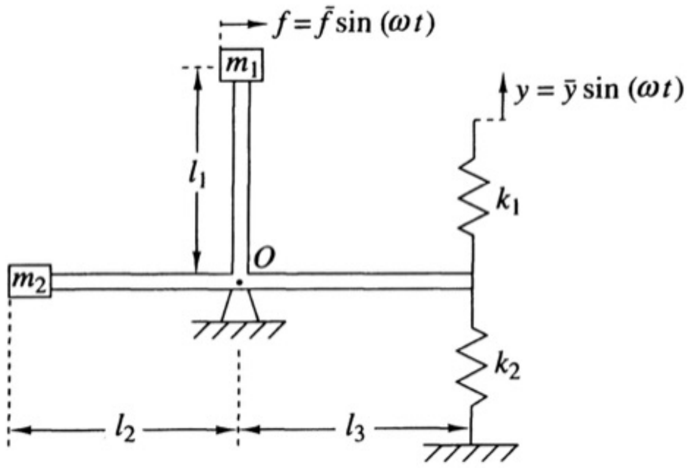
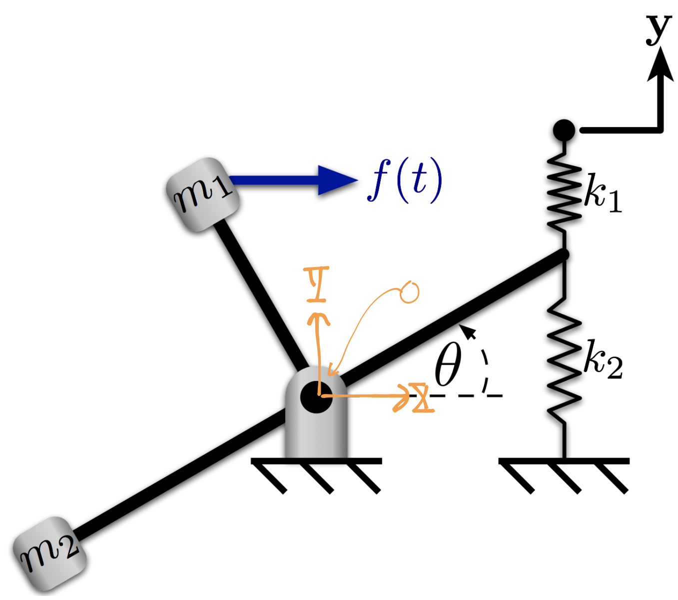
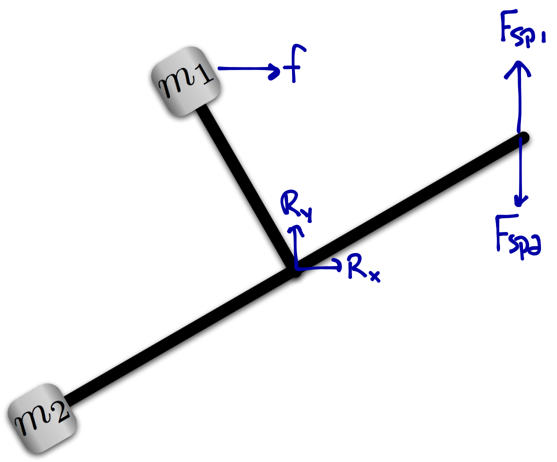


Figure P2.31





$F_{sp1} = k_1(y - l_3\sin\theta)$

$F_{sp2} = k_2(l_3\sin\theta)$

Sum moments about point O

$$\Sigma \bar{M}_o = \bar{r}_{sp/o} \times \left(k_1(y - l_3\sin\theta) - k_2(l_3\sin\theta)\right)\bar{J} + \bar{r}_{m_1/o} \times f\bar{I}$$

$$I_o\bar{\alpha} = (l_3\cos\theta\,\bar{I} + l_3\sin\theta\,\bar{J}) \times \left(k_1 y - (k_1+k_2)l_3\sin\theta\right)\bar{J} + (-l_1\sin\theta\,\bar{I} + l_1\cos\theta\,\bar{J}) \times f\bar{I}$$

$$I_o\ddot{\theta}\bar{K} = l_3\cos\theta\left(k_1 y - (k_1+k_2)l_3\sin\theta\right)\bar{K} - fl_1\cos\theta\,\bar{K}$$

$$I_o\ddot{\theta} = k_1 l_3 y\cos\theta - (k_1+k_2)l_3^2\sin\theta\cos\theta - fl_1\cos\theta$$ ← Assume small angles so $\sin\theta \doteq 0$, $\cos\theta \doteq 1$

$$I_o\ddot{\theta} = k_1 l_3 y - (k_1+k_2)l_3^2\theta - fl_1$$ ← $I_o = m_1 l_1^2 + m_2 l_2^2$

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} = k_1 l_3 y - (k_1+k_2)l_3^2\theta - fl_1$$ ← Equation of motion

$$\ddot{\theta} + \underbrace{\frac{(k_1+k_2)l_3^2}{m_1 l_1^2 + m_2 l_2^2}}_{=\omega_n^2}\theta = \underbrace{k_1 l_3 y - fl_1}$$ ← Linearized equation of motion

For there to be no motion, this term must be zero

$$k_1 l_3 y = fl_1 \rightarrow \boxed{f = \frac{k_1 l_3 y}{l_1} = \frac{k_1 l_3}{l_1}\bar{y}\sin\omega t}$$

Problem 2.39

2.39. Find the transfer function of support excitation y to response angle θ for the pendular system shown in Figure P2.38. Make sure to linearize your equations. The pendulum is of length l and the freely pivoted upper end of the pendulum is moved horizontally according to

$$y(t) = a\sin(\omega t)$$

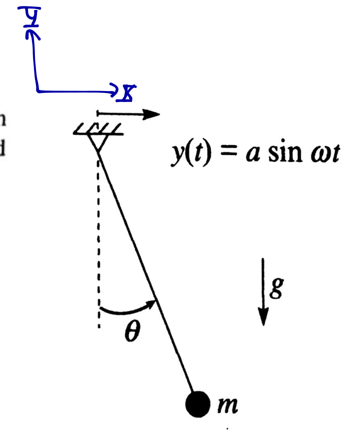


Figure P2.39

I will get the equations of motion using Lagrange's Method.

First, define the position of m

$$r_{m/o} = (y + l\sin\theta)\bar{I} - l\cos\theta\bar{J}$$

Using this write the velocity of m

$$\dot{r}_{m/o} = (\dot{y} + l\dot{\theta}\cos\theta)\bar{I} + l\dot{\theta}\sin\theta\bar{J} = \bar{v}_m$$

So,

$$T = \frac{1}{2}m v_m^T v_m = \frac{1}{2}m\left[(\dot{y} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right]\cdot\left[(\dot{y} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right]$$

$$= \frac{1}{2}m\left[(\dot{y} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right] = \frac{1}{2}m\left[\dot{y}^2 + 2l\dot{y}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right]$$

$$= \frac{1}{2}m\left[\dot{y}^2 + 2l\dot{y}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right]$$

$$V = -mgl\cos\theta$$

$$L = \frac{1}{2}m\left[\dot{y}^2 + 2l\dot{y}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right] + mgl\cos\theta \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left(2l\dot{y}\cos\theta + 2l^2\dot{\theta}\right) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\ddot{y}\cos\theta - ml\dot{y}\dot{\theta}\sin\theta + ml^2\ddot{\theta}$$

$$= ml^2\ddot{\theta} - ml\dot{y}\dot{\theta}\sin\theta + ml\ddot{y}\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{y}\dot{\theta}\sin\theta - mgl\sin\theta$$

$$ml^2\ddot{\theta} + ml\ddot{y}\cos\theta + mgl\sin\theta = 0 \longrightarrow$$ Assume small angles so $\sin\theta \approx \theta$ and $\cos\theta \approx 1$

$$ml^2\ddot{\theta} + ml\ddot{y} + mgl\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\theta = -\frac{1}{l}\ddot{y} \longleftarrow$$ Linearized Equation of motion with $\omega_n = \sqrt{\frac{g}{l}}$

Now, assume $y(t) = a\sin\omega t$ so $\ddot{y}(t) = -\omega^2 a\sin\omega t$

assume $x(t) = \bar{x}\sin\omega t$ $\quad(\ddot{x} = -\omega^2\bar{x}\sin\omega t)$

Plug into the equation of motion $\rightarrow (-\omega^2 + \omega_n^2)\bar{x}\cancel{\sin\omega t} = \frac{\omega^2}{l}a\cancel{\sin\omega t} \rightarrow \boxed{\frac{\bar{x}}{a} = \frac{\omega^2}{l(\omega_n^2 - \omega^2)}}$

Problem 2.42

2.42. Consider the pendular system illustrated in Figure P2.42 (lumped mass m on the end of a rigid rod of length l). A small motor at O produces a sinusoidally varying torque ($M = \bar{M}\sin(\omega t)$) that acts on the pendulum. Find the transfer function from input torque to response angle $\theta(t)$. Linearize your system equations about $\theta = 0$.

We set this one up in one of the first lectures.

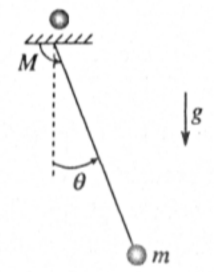
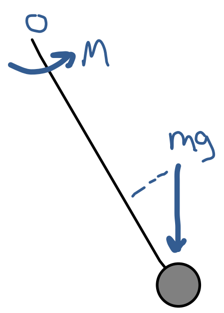


Figure P2.42

FBD



Sum moments about point O

$$I\ddot{\theta} = \sum \bar{M}_o = -mgl\sin\theta + M$$

$$ml^2\ddot{\theta} + mgl\sin\theta = M \quad \leftarrow \text{assume small angles}$$

$$ml^2\ddot{\theta} + mgl\theta = M$$

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{M}{ml^2}$$

Assume $\theta(t) = \bar{\theta}\sin(\omega t)$ ← match the form of the input

$$\dot{\theta}(t) = \omega\bar{\theta}\cos(\omega t)$$

$$\ddot{\theta}(t) = -\omega^2\bar{\theta}\sin(\omega t)$$

plug these back into the Eq. of Motion

$$-\omega^2\bar{\theta}\sin\omega t + \underbrace{\frac{g}{l}}_{\omega_n^2}\bar{\theta}\sin\omega t = \frac{\bar{M}}{ml^2}\sin\omega t$$

$$(-\omega^2 + \omega_n^2)\bar{\theta}\cancel{\sin\omega t} = \frac{\bar{M}}{ml^2}\cancel{\sin\omega t}$$

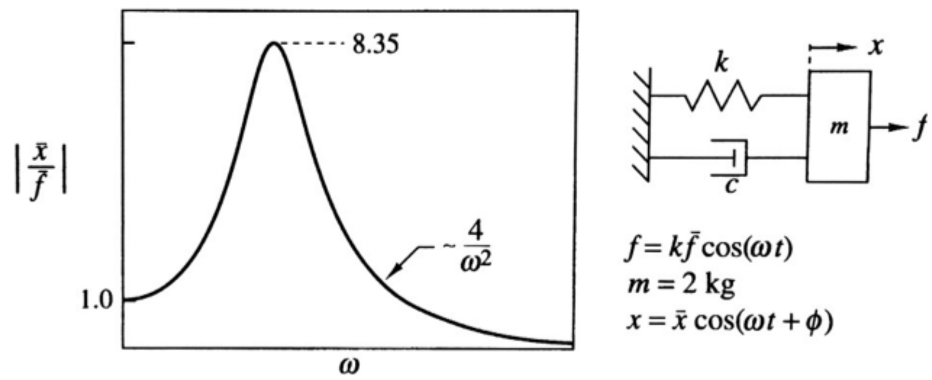
$$\boxed{\frac{\bar{\theta}}{\bar{M}} = \frac{1/ml^2}{\omega_n^2 - \omega^2}} \quad \longleftarrow$$

Notice that this has the same form as the direct force linear response.

You could have recognized that and jumped directly here.

Problem 2.43

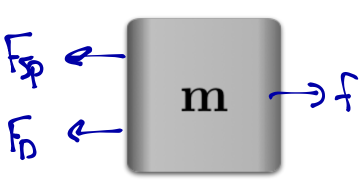
2.43. Consider the spring-mass-damper system illustrated in Figure P2.43. You need to determine the actual values for k and c but have only a limited amount of information. Given the data shown, determine k and c.



$f = k\bar{f}\cos(\omega t)$
$m = 2$ kg
$x = \bar{x}\cos(\omega t + \phi)$

Figure P2.43

FBD



$$m\ddot{x} = -kx - c\dot{x} + f$$
$$m\ddot{x} + c\dot{x} + kx = f$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f}{m}$$

$f = k\bar{f}\cos\omega t$ assume $x(t) = \bar{x}\cos(\omega t + \phi)$

If we blindly follow the suggestion, we get this, which is not easy to solve.

$$(-\omega^2 + \omega_n^2)\cos(\omega t + \phi) - (2\zeta\omega\omega_n)\sin(\omega t + \phi) = \frac{k}{m}\bar{f}\cos\omega t$$

Instead, rewrite $f(t)$ in complex form, knowing that we'll want the real part of the response:

$f(t) = k\bar{f}e^{i\omega t}$ so assume $x(t) = \bar{x}e^{i\omega t}$

$$(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2)\bar{x}\cancel{e^{i\omega t}} = \frac{k}{m}\bar{f}\cancel{e^{i\omega t}}$$

$$\frac{\bar{x}}{\bar{f}} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + (2i\zeta\omega\omega_n)}$$

Write this is magnitude and phase form:

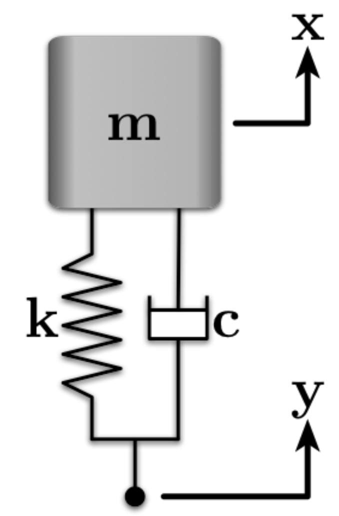
$$\left|\frac{\bar{x}}{\bar{f}}\right| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

Estimate that the peak occurs at $\omega = \omega_n$ (Note: It is actually slightly shifted for damped system)

at $\omega = \omega_n$

$$\left|\frac{\bar{x}}{\bar{f}}\right| = \frac{\omega_n^2}{2\zeta\omega_n^2} = \frac{1}{2\zeta} = 8.35 \quad \text{so} \quad \boxed{\zeta \approx 0.06}$$

Problems 2.62 - 2.65



2.62. Consider again the seismically excited, damped SDOF system Figure P2.1. What is the magnitude of m's response to a ground excitation of $y(t) = .01\cos(1.25\,t)$ for $k = 12,000$ N/m, $m = 20$ kg, and $\zeta = .001$? This should give you a general feel for how very lightly damped structures would respond in an earthquake.

2.63. What is the magnitude of the acceleration response for the system shown in Figure P2.1 for $m = 1$ kg, $c = 10$ N·s/m, $k = 100$ N/m, and $y = .005\cos(50\,t)$?

2.64. Repeat Problem 2.63 and then re-solve for two different cases. In case 1, let $k = 0$ and in case 2, let $c = 0$. Which is dominant, c or k, in determining the actual acceleration response magnitude?

2.65. Repeat Problem 2.62 but use $\zeta = .01$. Does this improve the situation much? } See IPython Notebook.

We know the equation of motion for this system is:

$$m\ddot{x}+c\dot{x}+kx = c\dot{y}+ky \qquad \text{or} \qquad \ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = 2\zeta\omega_n\dot{y}+\omega_n^2 y$$

We can also write the transfer function of this system as:

$$G(\omega)=\frac{\bar{X}}{\bar{Y}} = \sqrt{\frac{\omega_n^4+(2\zeta\omega\omega_n)^2}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}\; e^{i\phi} \qquad \text{where } \phi = \tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right) - \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)$$

Problem 2.62

From the parameters given $\omega_n=\sqrt{\frac{k}{m}}=\sqrt{\frac{12000}{20}}=\sqrt{600}$ and $\zeta=0.01$

The amplitude of the response is $G(\omega)\,\bar{y} \cong 0.01\,m$ ← The input is passed nearly directly to the output

{ See the IPython Notebook for details.

Problem 2.63

This is nearly identical to 2.62, but with different parameters. The only "trick" is to recognize that the magnitude of the acceleration response is just ω^2 times the magnitude of the position response.

$$\text{if - } x(t)=\bar{x}\cos(\omega t) \quad \text{then} \quad \ddot{x}(t)=-\omega^2\bar{x}\cos(\omega t)$$

See the IPython notebook for the numbers.

Problem 2.64

$k=0$ case: $m\ddot{x}+c\dot{x}=c\dot{y}$ Assume $y(t)=\bar{y}e^{i\omega t}$ so $x(t)=\bar{x}e^{i\omega t}$

$$(-m\omega^2+ic\omega)\bar{x}\cancel{e^{i\omega t}} = ic\omega\bar{y}\cancel{e^{i\omega t}}$$

$$\frac{\bar{x}}{\bar{y}} = \frac{ic\omega}{-m\omega^2-ic\omega} \rightarrow \frac{\sqrt{(c\omega)^2}\;e^{i\phi_1}}{\sqrt{(-m\omega^2)^2+(c\omega)^2}\;e^{i\phi_2}} \qquad \phi_1=\tan^{-1}(c\omega) \quad \text{and} \quad \phi_2=\tan^{-1}\left(\frac{c}{-m\omega}\right)$$

$c=0$ case is identical to 2.63 with different parameters.

} See IPython Notebook for more

Problem 2.67

2.67. Show that all the amplitude response plots for varying ζ go through the point $(\frac{\bar{x}}{\bar{y}}) = 1.0$ at $\Omega = \sqrt{2}$ for a seismically excited, damped SDOF system.

We know the transfer function for these systems is:

$$G(\Omega) = \sqrt{\frac{1 + (2\zeta\Omega)^2}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}}\, e^{i\phi}$$

Let's just look at its magnitude at $\Omega = \sqrt{2}$

$$G(\sqrt{2}) = \sqrt{\frac{1 + 8\zeta^2}{(1-2)^2 + 8\zeta^2}} = \sqrt{\frac{1+8\zeta^2}{1+8\zeta^2}} = 1$$