Problem 1.67

1.67. Find the linearized equation of motion about the equilibrium position of the system illustrated in Figure P1.67. You don't have to solve for the equilibrium position itself—just show what you'd need to do to obtain it. The torsional spring is uncompressed when $\theta = 0$. What is the linearized natural frequency?

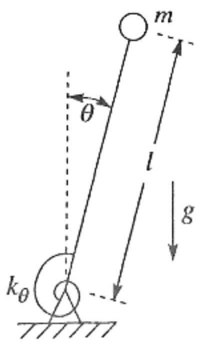
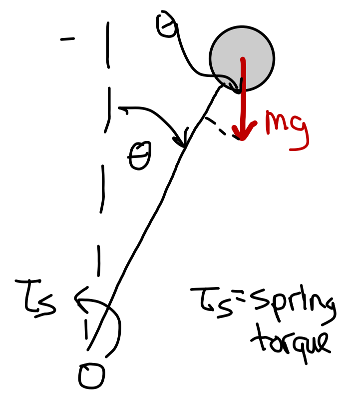


Figure P1.67

FBD



Sum moments about point O

$$I_O\ddot{\theta} = \circlearrowleft^{+}\sum M_O = \tau_s + mgl\sin\theta$$

$\tau_s = -k_\theta \theta$ (neg. sign because it "wants" to make $\theta = 0$)

$$ml^2\ddot{\theta} = -k_\theta\theta + mgl\sin\theta$$

$$ml^2\ddot{\theta} + k_\theta\theta - mgl\sin\theta = 0$$

define:

θ_{eq} – equil angle $(\neq 0)$

ϕ – angular deviation from θ_{eq}

so – $\theta = \theta_{eq} + \phi$ $\quad (\dot{\theta} = \dot{\phi}$ and $\ddot{\theta} = \ddot{\phi}$ because θ_{eq} is constant$)$

$$ml^2\ddot{\phi} + k_\theta(\theta_{eq} + \phi) - mgl\sin(\theta_{eq} + \phi) = 0$$

$$\sin(\theta_{eq} + \phi) = \sin\theta_{eq}\cos\phi + \cos\theta_{eq}\sin\phi \leftarrow \text{assume small angles } (\cos\phi \simeq 1,\ \sin\phi \simeq \phi)$$

$$= \sin\theta_{eq} + \phi\cos\theta_{eq}$$

so

$$ml^2\ddot{\phi} + k_\theta(\theta_{eq} + \phi) - mgl(\sin\theta_{eq} + \phi\cos\theta_{eq}) = 0$$

$$ml^2\ddot{\phi} + (k_\theta - mgl\cos\theta_{eq})\phi + \underbrace{k_\theta\theta_{eq} - mgl\sin\theta_{eq}}_{\text{at equilibrium these are equal}} = 0$$

$$\ddot{\phi} + \left(\frac{k_\theta - mgl\cos\theta_{eq}}{ml^2}\right)\phi = 0$$

$$\boxed{\omega_n = \sqrt{\frac{k_\theta - mgl\cos(\theta_{eq})}{ml^2}}}$$

Problem 1.67 using Lagrange's Equations

1.67. Find the linearized equation of motion about the equilibrium position of the system illustrated in Figure P1.67. You don't have to solve for the equilibrium position itself—just show what you'd need to do to obtain it. The torsional spring is uncompressed when $\theta = 0$. What is the linearized natural frequency?

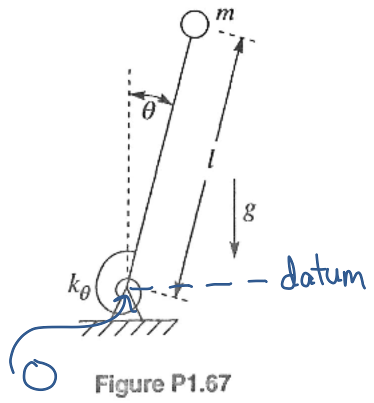
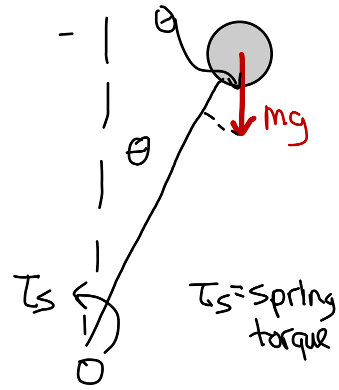


Figure P1.67

FBD



define our generalised coord. as $q_1 = \theta$

The mass in undergoing pure rotation so,

$$T = \frac{1}{2} I \omega^2 = \frac{1}{2}(ml^2)\dot{\theta}^2$$

If we define the gravity datum as point O, then

$$V_{gr} = mgh = mgl\cos\theta$$

The potential energy stored in the torsional spring is:

$$V_{sp} = \frac{1}{2} k_\theta \theta^2$$

$$\text{So } L = \underbrace{\left(\frac{1}{2} ml^2 \dot{\theta}^2\right)}_{T} - \underbrace{\left(mgl\cos\theta + \frac{1}{2} k_\theta \theta^2\right)}_{V}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2 \ddot{\theta}, \quad \frac{\partial L}{\partial \theta} = -\left(-mgl\sin\theta + k_\theta \theta\right)$$

$$ml^2 \ddot{\theta} + \left(-mgl\sin\theta + k_\theta \theta\right) = 0$$

$$ml^2 \ddot{\theta} + k_\theta \theta - mgl\sin\theta = 0 \longleftarrow \text{Same as using Newton/Euler}$$

From here, problem is identical

Problem 1.76

1.76. What is the natural frequency for the system shown in Figure P1.76?

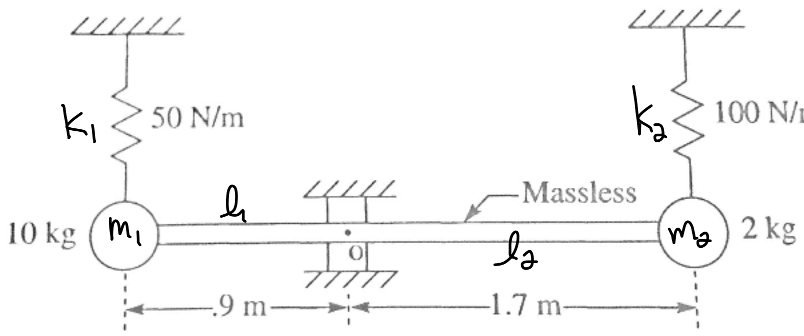
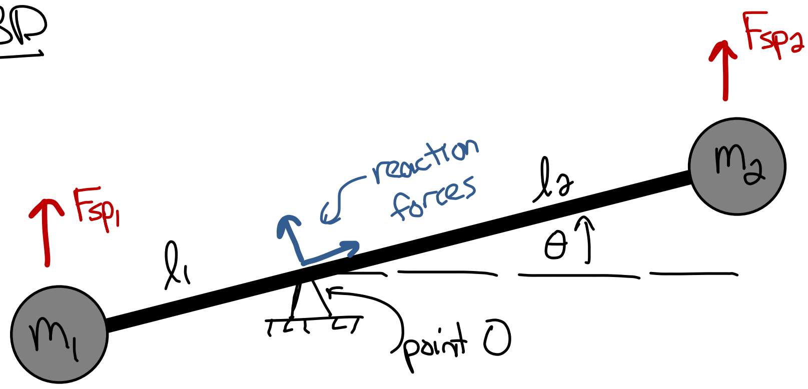


Figure P1.76

FBD



If we assume we are operating around equilibrium, we can ignore gravity.

Sum moments about O

$$I_{bar}\ddot{\theta} = (\bar{r}_{m_2/O} \times \bar{F}_{sp2}) + (\bar{r}_{m_1/O} \times \bar{F}_{sp1}) \longleftarrow$$

If we assume small angles, then $\bar{r}$ and $\bar{F}$ are $\perp$ in both cases so,

$$I_{bar}\ddot{\theta} = l_2 F_{sp2} - l_1 F_{sp1}$$

$$F_{sp1} = k_1 l_1 \theta \qquad F_{sp2} = k_2 l_2 \theta$$

$$I_{bar}\ddot{\theta} = -k_2 l_2^2 \theta - k_1 l_1^2 \theta$$

$$I_{bar} = m_1 l_1^2 + m_2 l_2^2$$

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} = -(k_2 l_2^2 + k_1 l_1^2)\theta$$

$$\ddot{\theta} + \left(\frac{k_1 l_1^2 + k_2 l_2^2}{m_1 l_1^2 + m_2 l_2^2}\right)\theta = 0$$

so $$\boxed{\omega_n = \sqrt{\frac{k_1 l_1^2 + k_2 l_2^2}{m_1 l_1^2 + m_2 l_2^2}}}$$

plug in values from problem to get

$$\boxed{\omega_n \cong 4.87}$$

Problem 1.76 using Lagrange's Equations

1.76. What is the natural frequency for the system shown in Figure P1.76?

define $q_1 = \theta$

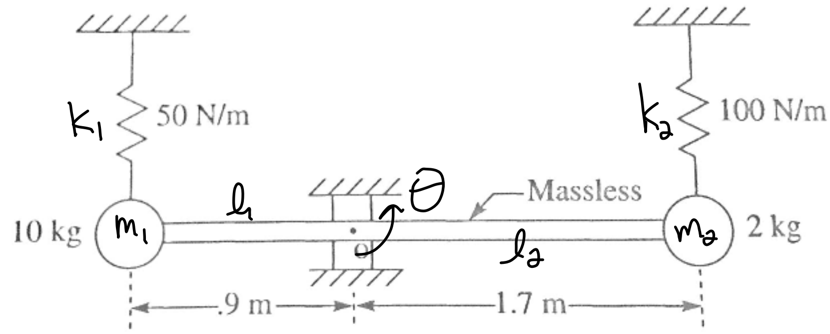


Figure P1.76

bar is in pure rotation so,

$$T = \frac{1}{2} I \omega^2 = \frac{1}{2} I \dot{\theta}^2 \qquad I = m_1 l_1^2 + m_2 l_2^2$$

We can ignore gravity, but we still have spring potential

$$V_{sp} = \frac{1}{2} k_1 (l_1 \theta)^2 + \frac{1}{2} k_2 (l_2 \theta_2)^2$$

So,

$$L = \frac{1}{2}(m_1 l_1^2 + m_2 l_2^2)\dot{\theta}^2 - \left(\frac{1}{2} k_1 l_1^2 \theta^2 + \frac{1}{2} k_2 l_2^2 \theta^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = (m_1 l_1^2 + m_2 l_2^2)\dot{\theta} \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (m_1 l_1^2 + m_2 l_2^2)\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\left(k_1 l_1^2 \theta + k_2 l_2^2 \theta\right)$$

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\theta = 0$$

$$\ddot{\theta} + \frac{k_1 l_1^2 + k_2 l_2^2}{m_1 l_1^2 + m_2 l_2^2}\theta = 0$$

Problem 1.105

1.105. Find the equations of motion for the system illustrated in Figure P1.105.

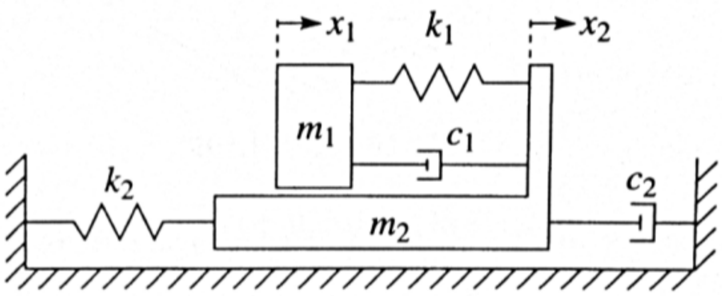
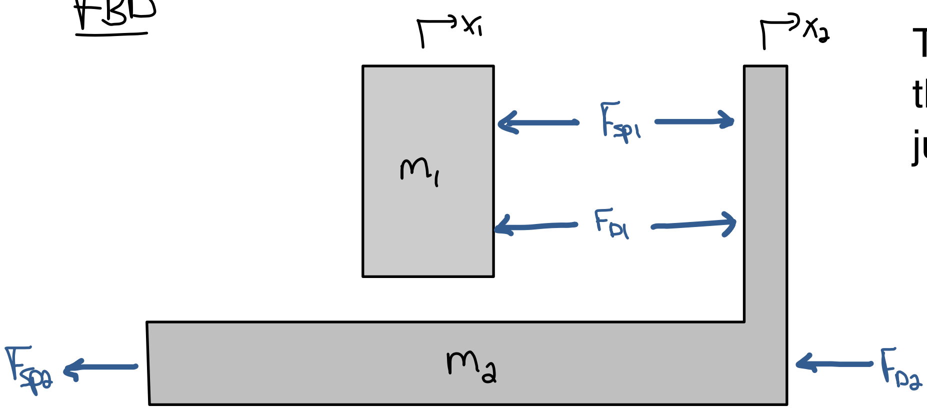


Figure P1.105

FBD



To get the equations of motion for this system using Newton/Euler, just apply $\sum \bar{F} = m\bar{a}$ for each mass.

$F_{sp1} = k_1\delta_1 \Rightarrow \delta_1$ is the difference between x_1 and x_2

$= k_1(x_1 - x_2)$

$F_{D1} = c_1\dot{\delta}_1 = c_1(\dot{x}_1 - \dot{x}_2)$

$F_{sp2} = k_2\delta_2$ – δ_2 is just x_2

$= k_2x_2$

$F_{D2} = c_2\dot{x}_2$

So, for m_1

$$m_1\ddot{x}_1 = -k_1(x_1 - x_2) - c_1(\dot{x}_1 - \dot{x}_2)$$

$$\boxed{m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = c_1\dot{x}_2 + k_1x_2}$$

For m_2

$$m_2\ddot{x}_2 = k_1(x_1 - x_2) + c_1(\dot{x}_1 - \dot{x}_2) - c_2\dot{x}_2 - k_2\dot{x}_2$$

$$\boxed{m_2\ddot{x}_2 + (c_1 + c_2)\dot{x}_2 + (k_1 + k_2)\dot{x}_2 = c_1\dot{x}_1 + k_1x_1}$$

Problem 1.105 using Lagrange's Method

1.105. Find the equations of motion for the system illustrated in Figure P1.105.

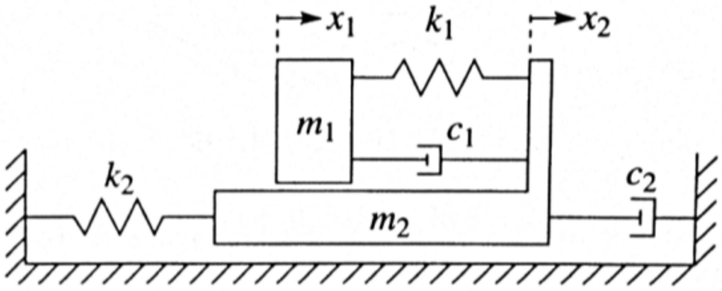
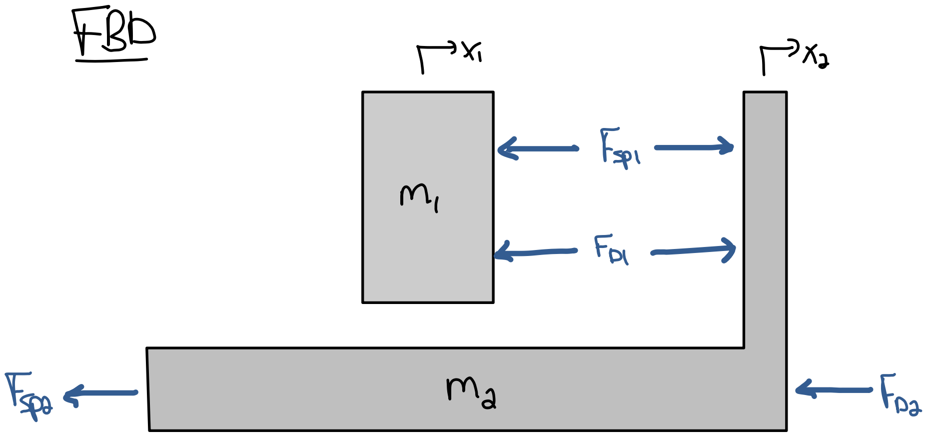


Figure P1.105

FBD



Even if you decide to solve via Lagrange's Method, it's still good practice to draw a Free Body Diagram.

Define our generalized coordinates as

$$\bar{q} = (x_1, x_2)$$

$F_{sp1} = k_1\delta_1 \Rightarrow \delta_1$ is the difference between x_1 and x_2

$= k_1(x_1 - x_2)$

$F_{D1} = c_1\dot{\delta}_1 = c_1(\dot{x}_1 - \dot{x}_2)$

$F_{sp2} = k_2\delta_2$ – δ_2 is just x_2

$= k_2 x_2$

$F_{D2} = c_2\dot{x}_2$

These same spring and damper deflections can be used to form the necessary energies for Lagrange's Method.

$$V_{sp1} = \frac{1}{2}k_1\delta_1^2 = \frac{1}{2}k_1(x_1 - x_2)^2$$

$$V_{sp2} = \frac{1}{2}k_2\delta_2^2 = \frac{1}{2}k_2 x_2^2$$

$$V_{TOT} = V_{sp1} + V_{sp2} = \frac{1}{2}k_1(x_1 - x_2)^2 + \frac{1}{2}k_2 x_2^2$$

$$RD_1 = \frac{1}{2}c_1\dot{\delta}_1^2 = \frac{1}{2}c_1(\dot{x}_1 - \dot{x}_2)^2$$

$$RD_2 = \frac{1}{2}c_2\dot{\delta}_2^2 = \frac{1}{2}c_2\dot{x}_2^2$$

$$RD_{TOT} = \frac{1}{2}c_1(\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2}c_2\dot{x}_2^2$$

$T_{m1} = \frac{1}{2}m_1\dot{x}_1^2$ $\quad$ $T_{m2} = \frac{1}{2}m_2\dot{x}_2^2$ $\quad$ $T_{TOT} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$

So, $L = T_{TOT} - V_{TOT} = \left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2\right) - \left(\frac{1}{2}k_1(x_1 - x_2)^2 + \frac{1}{2}k_2 x_2^2\right)$

For x_1 and x_2, apply Lagrange's Equation (with the damping term):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial RD}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

(0 because there are no external forces)

Problem 1.105 using Lagrange's Method (cont.)

$$L = T_{TOT} - V_{TOT} = \left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2\right) - \left(\frac{1}{2}k_1(x_1-x_2)^2 + \frac{1}{2}k_2x_2^2\right) \qquad RD = \frac{1}{2}c_1(\dot{x}_1-\dot{x}_2)^2 + \frac{1}{2}c_2\dot{x}_2^2$$

For x_1

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) + \frac{\partial RD}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1, \quad \frac{\partial RD}{\partial \dot{x}_1} = c_1(\dot{x}_1-\dot{x}_2), \quad \frac{\partial L}{\partial x_1} = -k_1(x_1-x_2)$$

$$m_1\ddot{x}_1 + c_1(\dot{x}_1-\dot{x}_2) - (-k_1(x_1-x_2)) = 0$$

$$\boxed{m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = c_1\dot{x}_2 + k_1x_2}$$

For x_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) + \frac{\partial RD}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2, \quad \frac{\partial RD}{\partial \dot{x}_2} = -c_1(\dot{x}_1-\dot{x}_2) + c_2\dot{x}_2, \quad \frac{\partial L}{\partial x_2} = k_1(x_1-x_2) - k_2x_2$$

$$m_2\ddot{x}_2 + \left[-c_1(\dot{x}_1-\dot{x}_2) + c_2\dot{x}_2\right] - \left[k_1(x_1-x_2) - k_2x_2\right] = 0$$

$$\boxed{m_2\ddot{x}_2 + (c_1+c_2)\dot{x}_2 + (k_1+k_2)x_2 = c_1\dot{x}_1 + k_1x_1}$$

Problem 2.10

2.10. The spring-mass system shown in Figure P2.8 experiences a 4 mm out-of- phase oscillation when the base is oscillated at 200 rad/s with an amplitude of 1.3 mm. What is the static deflection of the 5 kg mass?

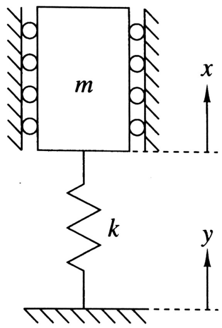


Figure P2.8

For this system we know the transfer function is:

$$G(\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2}$$

We can use this and the magnitude/phase information given to find ω_n. Then from ω_n, we can find k and determine the static deflection.

out of phase

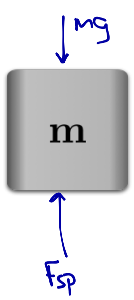
$$\frac{X}{Y} = \frac{-4}{1.3} = \frac{\omega_n^2}{\omega_n^2 - (200)^2} \longrightarrow \frac{-4}{1.3}\left(\omega_n^2 - (200)^2\right) = \omega_n^2$$

$$\frac{4}{1.3}(200)^2 = \left(1 + \frac{4}{1.3}\right)\omega_n^2$$

$$\omega_n^2 = \frac{-4/1.3(200)^2}{1 + 4/1.3} = (173.75)^2$$

$$\omega_n = 173.75 = \sqrt{\frac{K}{M}} \longrightarrow K = 30188.68M \approx 150943.4$$

At static deflection the spring and gravitational forces balance



$$m\ddot{x} = F_{sp} - mg$$

$$\cancel{m\ddot{x}}^{\,0} = k(\cancel{y}^{\,0} - x) - mg \longleftarrow \text{At static equil } \ddot{x} = 0 \text{ and } y = 0$$

$$0 = -kx - mg$$

$$kx = mg$$

$$\boxed{x = \frac{mg}{k} = 0.0003m}$$

Problem 2.41

2.41. Find the transfer function between the input displacement y and the output force against the wall for the system shown in Figure P2.41. $y = \bar{y}\sin(\omega t)$

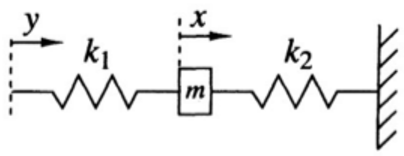
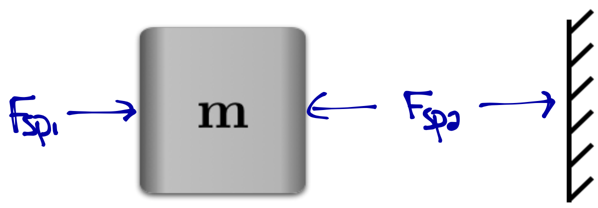


Figure P2.41



$$m\ddot{x} = k_1(y-x) + k_2 x$$

$$\ddot{x} + \frac{k_1+k_2}{m}x = \frac{k_1}{m}y \rightarrow \ddot{x} + \omega_n^2 x = \frac{k_1}{m}y \quad \text{where } \omega_n = \sqrt{\frac{k_1+k_2}{m}}$$

Assume a solution of $x(t) = \bar{X}\sin\omega t$ (to match the input)

$$(-\omega^2 + \omega_n^2)\bar{X}\sin\omega t = \frac{k_1}{m}\bar{Y}\sin\omega t$$

$$\underbrace{\bar{X} = \frac{k_1/m}{\omega_n^2-\omega^2}\bar{Y}}_{} \qquad \text{so } x(t) = \frac{k_1/m}{(\omega_n^2-\omega^2)}\bar{Y}\sin\omega t$$

We can write the transfer function from input $\bar{Y}$ to position $\bar{X}$ using this

$$\frac{\bar{X}}{\bar{Y}} = \frac{k_1}{m(\omega_n^2-\omega^2)}$$

We know that the force on the wall is the spring force, so we can write the transfer function from $\bar{Y}$ to F_{sp2}

$$\boxed{\frac{k_2\bar{X}}{\bar{Y}} = \frac{k_2 k_1}{m(\omega_n^2-\omega^2)}}$$

If you didn't see that, you could work from $x(t)$

$$\frac{F_{sp2}}{Y} = \frac{k_2 x(t)}{y(t)} = \frac{k_2\bar{X}\cancel{\sin\omega t}}{\bar{Y}\cancel{\sin\omega t}} = \frac{k_2\bar{X}}{\bar{Y}}$$

Problem 2.56

2.56. Solve for the velocity response $\dot{x}$ of the system illustrated in Figure P2.56. $y = .02\sin(150t)$, $k_1 = 10,000$ N/m, $k_2 = 5000$ N/m, $m = .5$ kg, $c = 8.66$ N·s/m.

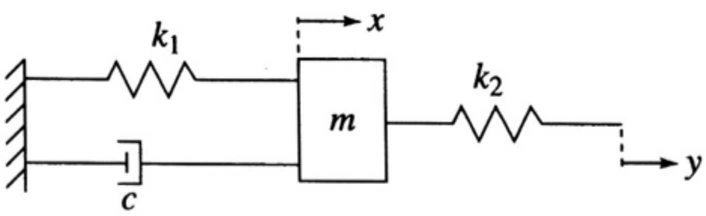
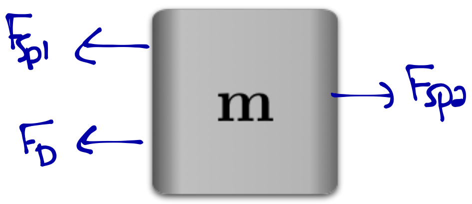


Figure P2.56

FBD



$$m\ddot{x} = -F_{sp1} - F_D + F_{sp2}$$

$$m\ddot{x} = -k_1 x - c\dot{x} + k_2(y - x)$$

$$m\ddot{x} + c\dot{x} + (k_1 + k_2)x = k_2 y \longrightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{k_2}{m} y \qquad \omega_n = \sqrt{\frac{k_1+k_2}{m}}$$

$$2\zeta\omega_n = \frac{c}{m}$$

Let $y(t) = \bar{y}e^{i\omega t}$ (Remembering that the "actual" $y(t)$ is the imaginary component of this)

Assume $x(t) = \bar{x}e^{i\omega t}$ to match.

Sub into the equations of motion.

$$\left(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2\right)\bar{x}\,\cancel{e^{i\omega t}} = \frac{k_2}{m}\bar{y}\,\cancel{e^{i\omega t}}$$

$$\bar{X} = \frac{k_2/m}{(\omega_n^2 - \omega^2) + (2i\zeta\omega\omega_n)}\bar{Y}$$

So

$$x(t) = \frac{k_2/m}{(\omega_n^2 - \omega^2) + (2i\zeta\omega\omega_n)}\bar{Y}e^{i\omega t}$$

We want the imaginary part of this, to match the input.

$$x(t) = \frac{k_2/m\left[(\omega_n^2 - \omega^2) - 2i\zeta\omega\omega_n\right]}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\bar{Y}(\cos\omega t + i\sin\omega t)$$

$$\bar{X}_r = \frac{k_2/m\,(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\bar{Y} \qquad \bar{X}_c = \frac{\frac{k_2}{m}(-2\zeta\omega\omega_n)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\bar{Y}$$

Remember that we can write the response like

$$x(t) = \bar{X}e^{i\omega t}$$

$$x(t) = (\bar{X}_r + i\bar{X}_c)(\cos\omega t + i\sin\omega t)$$

$$= (\bar{X}_r\cos\omega t - \bar{X}_c\sin\omega t) + i(\bar{X}_r\sin\omega t + \bar{X}_c\cos\omega t)$$

Problem 2.56 (cont.)

$$x(t) = (\bar{X}_r \cos\omega t - \bar{X}_c \sin\omega t) + i(\bar{X}_r \sin\omega t + \bar{X}_c \cos\omega t)$$ ← We want the imaginary part of this.

$$\bar{X}_r = \frac{k_2/m\,(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2}\,\bar{Y} \qquad \bar{X}_c = \frac{\frac{k_2}{m}(-2\zeta\omega\omega_n)}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2}\,\bar{Y}$$

$$x(t) = \bar{X}_r \sin\omega t + \bar{X}_c \cos\omega t$$

$$\left.\dot{x}(t) = \omega\bar{X}_r \cos\omega t - \omega\bar{X}_c \sin\omega t\right\}$$ Now, plug in the values from the problem

I used the IPython Notebook for the numerical part of this problem to find:

$$\boxed{\dot{x}(t) = 4\cos(150t) + 1.39\sin(150t)}$$