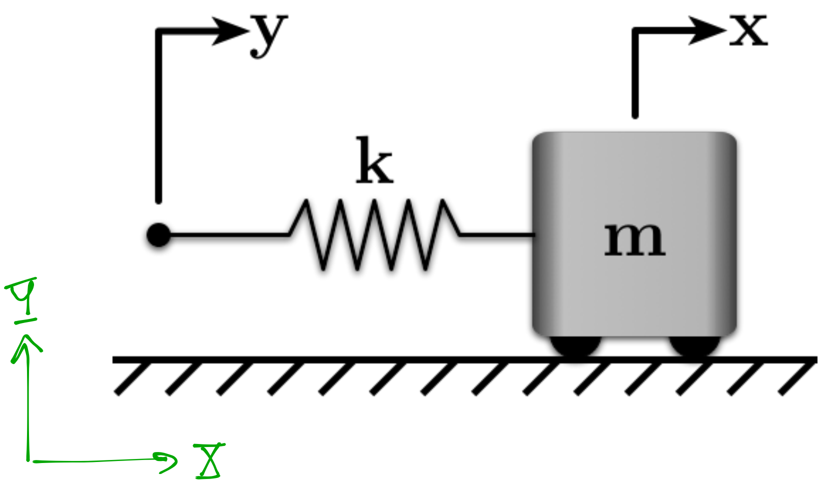
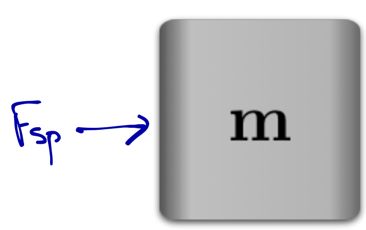
A Simple Example



FBD



$$m\ddot{x} = F_{sp} \rightarrow m\ddot{x} = k(y-x) \rightarrow m\ddot{x} + kx = ky \rightarrow \ddot{x} + \omega_n^2 x = \omega_n^2 y$$

The first time we solved this example, we assumed $y(t) = \bar{y}\sin\omega t$,

so the form of $x(t)$ was:

$$x(t) = a\sin\omega t + b\cos\omega t$$

Now, let's solve using the complex representation.

$$y(t) = \bar{y}\sin\omega t = \mathrm{Im}\left[\bar{y}e^{i\omega t}\right]$$

So, assume $x(t) = \bar{x}e^{i\omega t}$ $\left[\text{so } \ddot{x} = -\omega^2\bar{x}e^{i\omega t}\right]$

$$\ddot{x} + \omega_n^2 x = \omega_n^2 y \longrightarrow \left[-\omega^2\bar{x}\cancel{e^{i\omega t}}\right] + \omega_n^2\bar{x}\cancel{e^{i\omega t}} = \omega_n^2\bar{y}\cancel{e^{i\omega t}}$$

$$\left[\omega_n^2 - \omega^2\right]\bar{x} = \omega_n^2\bar{y}$$

$$\bar{x} = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y}$$

So, $\boxed{x(t) = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y}e^{i\omega t}}$

A Simple Example (cont.)

Q: But, the response has to be real. How do we get it?

Look at $x(t) = \bar{x} e^{i\omega t} = \bar{x}(\cos\omega t + i\sin\omega t)$

Let $\bar{x} = \bar{x}_r + i\bar{x}_i$ $\quad \bar{x}_r$ = real part of $\bar{x}$ and $\bar{x}_i$ = imaginary part of $\bar{x}$

So $x(t) = (\bar{x}_r + i\bar{x}_i)(\cos\omega t + i\sin\omega t)$

$= (\bar{x}_r \cos\omega t - \bar{x}_i \sin\omega t) + i(\bar{x}_r \sin\omega t + \bar{x}_i \cos\omega t)$ ← Take the real or imaginary part of this based on the form of the input

Here, $\bar{x} = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y}$

Q: What are $\bar{x}_r$ and $\bar{x}_i$ for this $\bar{x}$?

$$\bar{x} = \left[\frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y}\right] + i[0] \longrightarrow \bar{x}_r = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y} \quad \bar{x}_i = 0$$

Because $y(t) = \mathrm{Im}[\bar{y}e^{i\omega t}] = \bar{y}\sin\omega t$, we want the Imaginary part of $x(t)$

$$x(t) = (\bar{x}_r \cos\omega t - \bar{x}_i \sin\omega t) + i(\bar{x}_r \sin\omega t + \bar{x}_i \cos\omega t)$$

$$x(t) = x_r \sin\omega t + x_i \cos\omega t$$

$$\boxed{x(t) = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y}\sin\omega t}$$

← Look familiar? It should. It matches our previous solution

Problem 1.102

1.102. Find the equation of motion for the system illustrated in Figure P1.102.

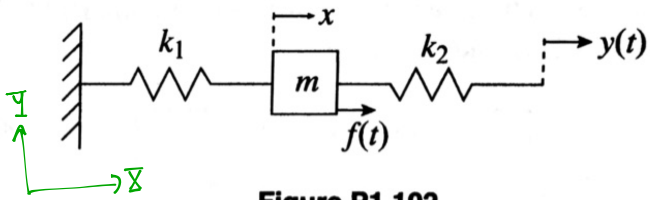
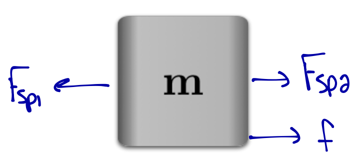


Figure P1.102

FBD



$$m\ddot{x} = -F_{sp1} + F_{sp2} + f$$

$$m\ddot{x} = -k_1 x + k_2(y-x) + f$$

$$\boxed{m\ddot{x} + (k_1+k_2)x = k_2 y + f}$$

Newton/Euler

Lagrange

$\bar{q} = (x) \leftarrow$ we'll treat $y(t)$ as an input

$$T = \frac{1}{2} m \dot{x}^2$$

$$V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2(y-x)^2 = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2(y^2 - 2xy + x^2) = \frac{1}{2}(k_1+k_2)x^2 - k_2 xy + \frac{1}{2}k_2 y^2$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k_1+k_2)x^2 + k_2 xy - \frac{1}{2}k y^2$$

We have an external, non-conservative force $f(t)$. So, we need to use virtual work to find Q_1 that results from it

$\delta W = \bar{F} \cdot \delta \bar{r}$ where $\bar{F}$ is the vector rep. of the force — $f\bar{I}$ in this case

$\delta \bar{r}$ is the virtual displacement of the point at which it's acting

$\nwarrow$ just $\delta x \bar{I}$ in this case

$$= f\bar{I} \cdot (\delta x \bar{I})$$

$$= f\,\delta x$$

Then, match the terms in this to $\delta W = Q_1 \delta q_1 = Q_1 \delta x \rightarrow Q_1 = f$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_1$$

Problem 1.102 (cont.)

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k_1+k_2)x^2 + k_2 xy - \frac{1}{2}ky^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -(k_1+k_2)x + k_2 y \qquad Q_1 = f$$

$$m\ddot{x} - \left[-(k_1+k_2)x + k_2 y\right] = f$$

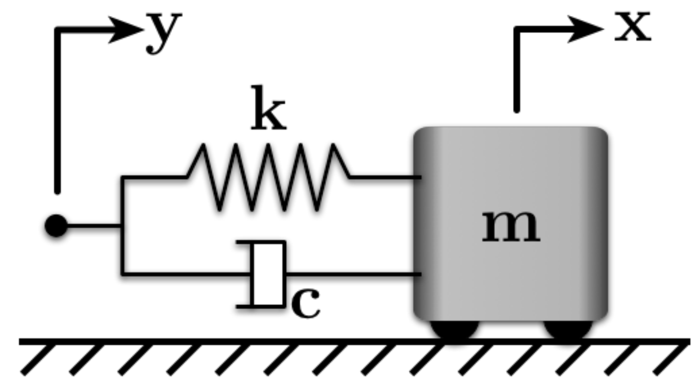
$$\boxed{m\ddot{x} + (k_1+k_2)x = k_2 y + f}$$ ← This matches the Newton/Euler Solution (It better!!!)

Q: What is the natural freq of this system?

Divide by m → $\ddot{x} + \underbrace{\left(\frac{k_1+k_2}{m}\right)}_{=\omega_n^2} x = \frac{k_2}{m}y + \frac{1}{m}f$

$= \omega_n^2 \rightarrow \omega_n = \sqrt{\frac{k_1+k_2}{m}}$

Position (Seismic) Harmonic Inputs with Damping (Sec. 2.8)



For the undamped case, the repsonse of this system was identical to the direct-forced case. With damping, that is no longer true.

We know that the equation of motion for this system is: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y$

Let $y(t) = \bar{y}e^{i\omega t}$ and assume $x(t) = \bar{x}e^{i\omega t}$ (to match the form of y)

$$\left(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2\right)\bar{x}\cancel{e^{i\omega t}} = \left(2i\zeta\omega\omega_n + \omega_n^2\right)\bar{y}\cancel{e^{i\omega t}}$$

$$G(\omega) = \frac{\bar{x}}{\bar{y}} = \frac{2i\zeta\omega\omega_n + \omega_n^2}{\omega_n^2 - \omega^2 + 2i\zeta\omega\omega_n}$$

} Write both the numerator and denominator in the $|\cdot|e^{i\phi}$ form

$$\boxed{G(\omega) = \frac{\sqrt{\omega_n^4 + (2\zeta\omega\omega_n)^2}\, e^{i\phi_1}}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\, e^{i\phi_2}}}$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2}\right) = \tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right)$$

$$\phi_2 = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

so

$$G(\omega) = \frac{\sqrt{\omega_n^4 + (2\zeta\omega\omega_n)^2}}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}}\, e^{i(\phi_1 - \phi_2)}$$

We can also normalize by using $\Omega = \frac{\omega}{\omega_n}$ to find:

$$\boxed{G(\Omega) = \sqrt{\frac{1 + (2\zeta\Omega)^2}{(1-\Omega^2)^2 + (2\zeta\Omega)^2}}\, e^{i\phi} \quad \text{where } \phi = \tan^{-1}(2\zeta\Omega) - \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)}$$

Position (Seismic) Harmonic Inputs with Damping (cont.)