Transfer Functions (Sec. 2.5)

Also a controls preview/review

Describe the relationship between the input and the output.

How is the input "transferred" to the output?

Usually, transfer functions are written as:

$\frac{\text{Output}}{\text{Input}}$ — from our previous examples $\frac{x(t)}{y(t)}$ or $\frac{x(t)}{f(t)}$ } but we usually don't write them in the time domain

Let's look at the undamped, seismically excited example

$x(t) = \frac{\omega_n^2}{\omega_n^2 - \omega^2} \bar{Y} \sin \omega t$ and $y(t) = \bar{Y} \sin \omega t$ so

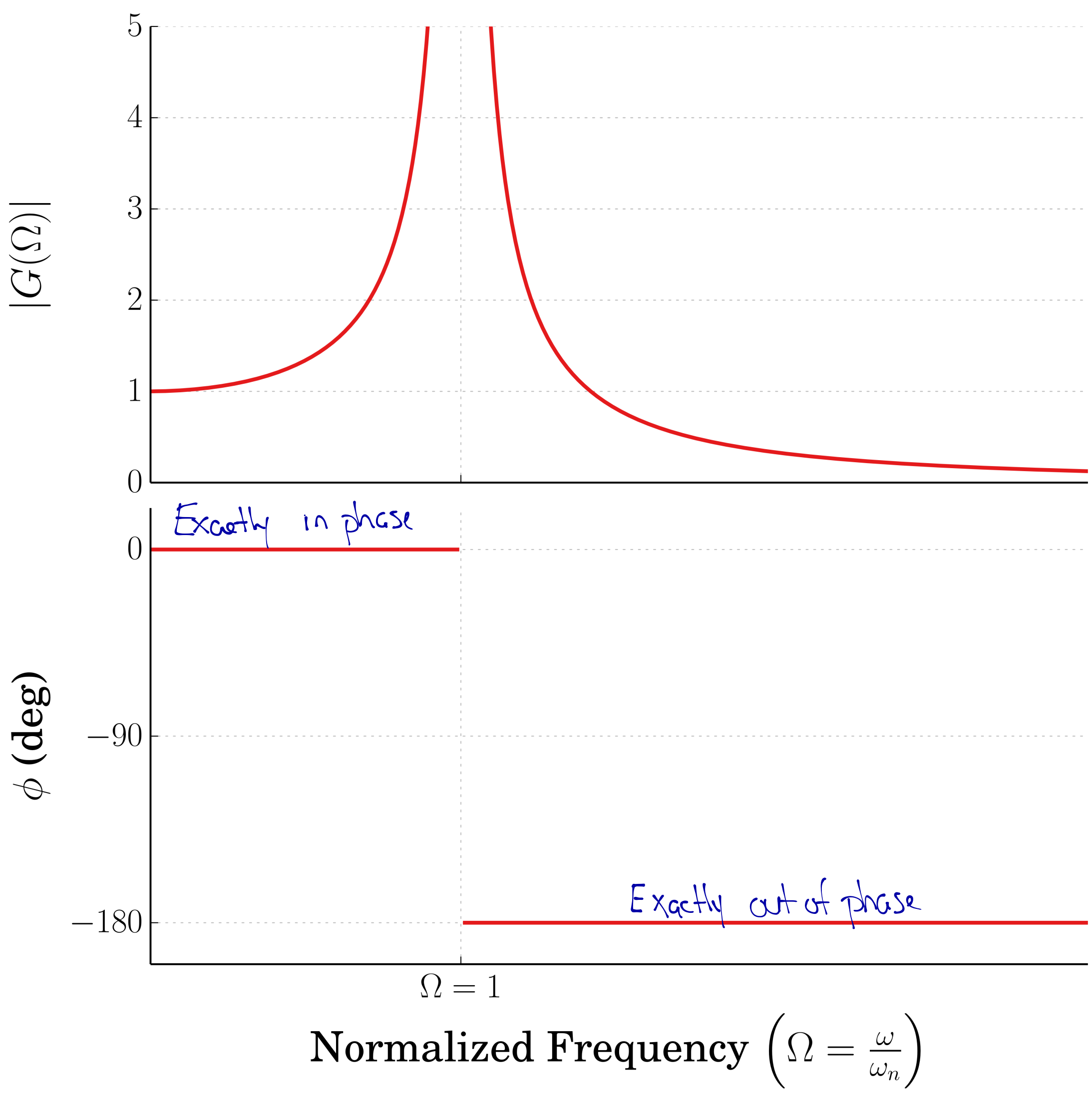
$$\frac{x(t)}{y(t)} = \frac{\frac{\omega_n^2}{\omega_n^2-\omega^2} \cancel{\bar{Y} \sin\omega t}}{\cancel{\bar{Y} \sin \omega t}} = \frac{\omega_n^2}{\omega_n^2 - \omega^2}$$

We often write transfer functions as $G(\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2}$ } Note: The book uses lower-case g. I'm sticking with the controls convention of caps

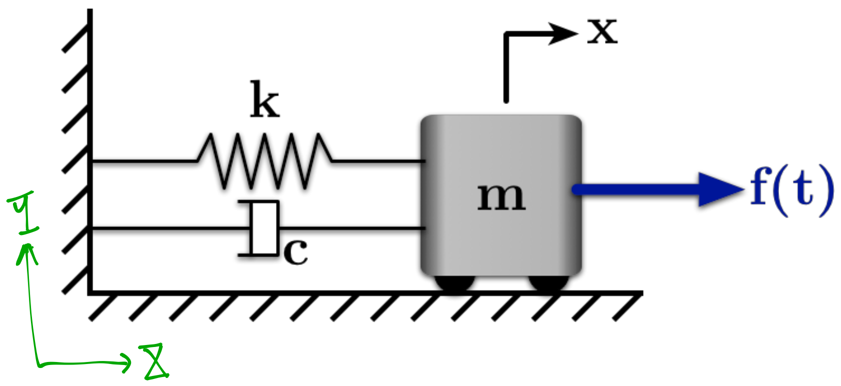
We can also normalize/nondimensionalize the freq.

$G(\Omega) = \frac{1}{1-\Omega^2}$ ← plotting this would match our earlier freq response plots. We were plotting the transfer function.

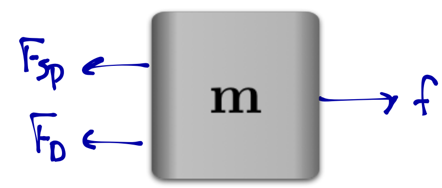
We generally separate magnitude and phase:



Forced Vibration with Viscous Damping (Sec. 2.6)



FBD



Damped equation of motion

$$m\ddot{x} = f - F_{sp} - F_D = f - kx - c\dot{x}$$

$$m\ddot{x} + c\dot{x} + kx = f \longrightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m} f \longleftarrow \text{Assume } f(t) = \bar{f}\sin\omega t$$

Like the undamped case, assume a solution of:

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

Plug into the eq. of motion and group sin and cos terms:

$$\left(-\omega^2 a + 2\zeta\omega_n\omega b + \omega_n^2 a\right)\cos\omega t + \left(-\omega^2 b - 2\zeta\omega_n\omega a + \omega_n^2 b\right)\sin\omega t = \frac{1}{m}\bar{f}\sin\omega t$$

Match sine and cosine terms:

$$\cos \rightarrow \quad -\omega^2 a + 2\zeta\omega_n\omega b + \omega_n^2 a = 0$$

$$\sin \rightarrow \quad -\omega^2 b - 2\zeta\omega_n\omega a + \omega_n^2 b = \bar{f}/m$$

Note:

1) With damping, the cosine term remains

2) $b = 0$ iff $\omega = \omega_n$ and a is finite at $\omega = \omega_n$ $\longleftarrow$ The response is always finite

BUT this solution is ugly

Q: How else could we write the response?

$$x(t) = |\bar{x}|\cos(\omega t - \phi) \quad \text{or} \quad x(t) = |\bar{x}|\sin(\omega t - \phi)$$

We can use either, but usually best to match the form of the input. Ex) If $f(t) = \bar{f}\sin\omega t$, then choose $x(t) = |\bar{x}|\sin(\omega t - \phi)$

Forced Vibration with Viscous Damping (cont.)

So, convert the "ugly" solution to this form

⋮ (Lot's of math. See the book)

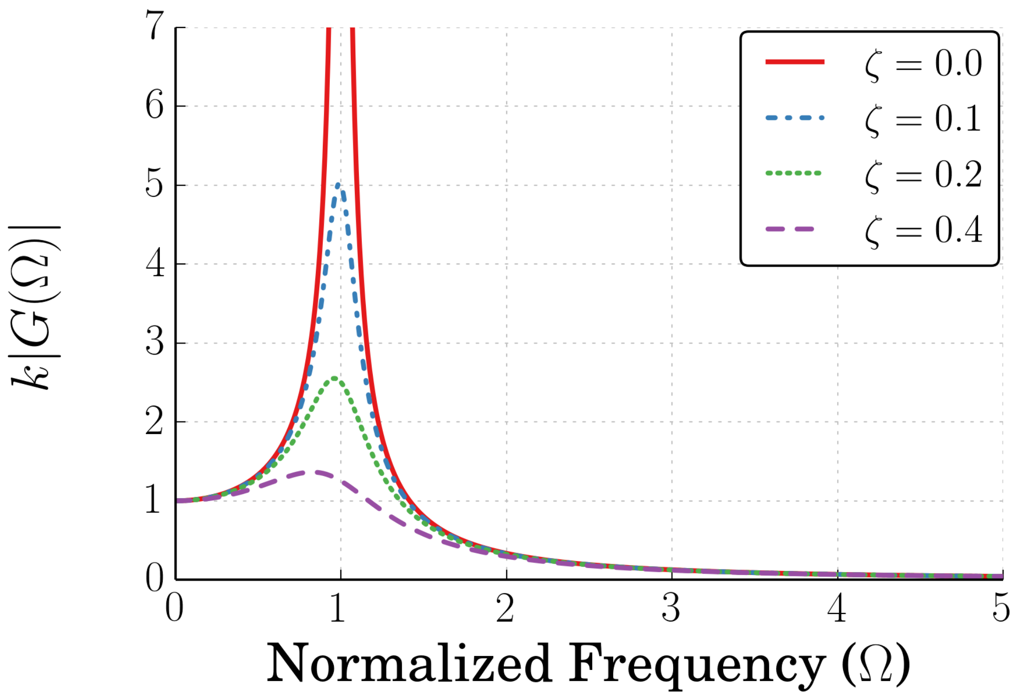
$$\underbrace{|\bar{X}| = \frac{\bar{F}}{m}\frac{1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}}$$ and $$\phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)$$

We can get the transfer function form of this (divide by $\bar{F}$)

$$|G(\omega)| = \frac{|\bar{X}|}{|\bar{F}|} = \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}$$

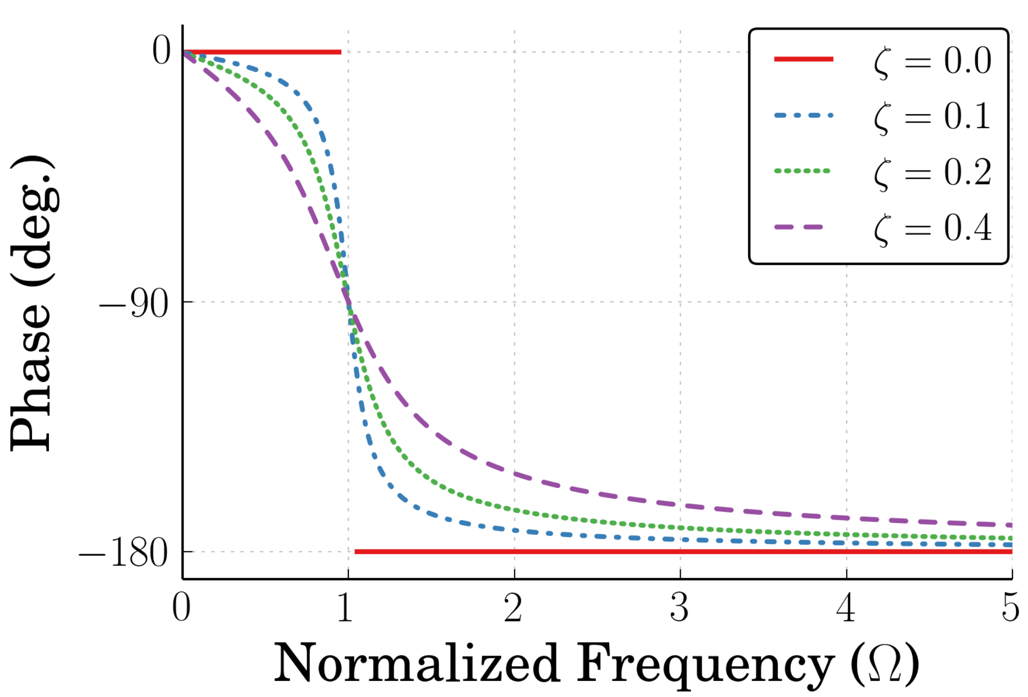
We can also normalize by $\Omega = \frac{\omega}{\omega_n} \longrightarrow$ $\underbrace{|G(\Omega)| = \frac{1}{k\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}}}$ and $\phi = \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)$

plot $kG(\Omega)$ to normalize the amplitude



Notice that damping limits the peak of the response at the natural frequency.

The trends as the input frequency goes to zero or infinity are the same for undamped and damped cases.



Notice that the damped responses transition from 0 to -180 deg phase shift.

Complex Representations (Sec. 2.7)

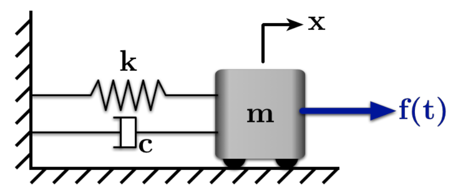
Instead of writing the response as sines and cosines, write:

$x(t) = \bar{X} e^{i\omega t}$, where $\bar{X}$ can be complex

Q: Why do this?

It makes the math (and some of the analysis, as a result) easier

Let's look at the 2nd-order, direct force, underdamped system This one



$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} f(t)$$

Let $f(t) = \bar{F} e^{i\omega t}$ and assume a solution of $x(t) = \bar{X} e^{i\omega t}$

$\left(\dot{x} = i\omega e^{i\omega t} \text{ and } \ddot{x} = -\omega^2 e^{i\omega t} \right)$

Plug into the equations of motion (just like the sine and cosine version):

$$\left(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2\right) \bar{X} \cancel{e^{i\omega t}} = \frac{1}{m} \bar{F} \cancel{e^{i\omega t}}$$

So,

$$\bar{X} = \frac{\bar{F}}{m} \left(\frac{1}{-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2} \right)$$

Now, let's divide both sides by $\bar{F}$ to get the transfer function

$$G(\omega) = \frac{\bar{X}}{\bar{F}} = \frac{1}{m\left(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2\right)}$$

BUT... this is complex???

Write it as $G(\omega) = |G(\omega)| e^{i\phi}$:

$$|G(\omega)| = \frac{1}{m\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_n\right)^2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2} \right)$$

Q: Does this look familiar? It should.

Complex? Can we have complex inputs or responses (in "real life")?

Look at $x(t) = \bar{X} e^{i\omega t} = \bar{X}(\cos\omega t + i \sin\omega t)$

Let $\bar{X} = \bar{X}_r + i\bar{X}_i$ $\quad$ $\bar{X}_r$ = real part of $\bar{X}$ and $\bar{X}_i$ = imaginary part of $\bar{X}$

So $x(t) = \left(\bar{X}_r + i\bar{X}_i\right)\left(\cos\omega t + i\sin\omega t\right)$

$$= \left(\bar{X}_r \cos\omega t - \bar{X}_i \sin\omega t\right) + i\left(\bar{X}_r \sin\omega t + \bar{X}_i \cos\omega t\right)$$

Take the real or imaginary part of this based on the form of the input

Complex Representations (cont.)

Q: What are $\bar{X}_r$ and $\bar{X}_c$?

$$\bar{X} = \frac{\bar{F}/m}{\omega_n^2 - \omega^2 + 2\zeta\omega\omega_n}$$

← mult by the complex conj. of the denominator $(\omega_n^2 - \omega^2 - 2i\zeta\omega\omega_n)$

Then the real part is $\bar{X}_r$ and the imaginary is $\bar{X}_c$

$$\bar{X}_r = \frac{\bar{F}/m\,(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \qquad \bar{X}_c = \frac{\bar{F}/m\,(2\zeta\omega\omega_n)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

Responses in the Complex Plane

$$x(t) = \bar{x}e^{i\omega t}$$

$$\dot{x}(t) = i\omega\bar{x}e^{i\omega t} \leftarrow i = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2} = e^{i\pi/2} \rightarrow \underbrace{\dot{x}(t) = \omega\bar{x}e^{i(\omega t + \frac{\pi}{2})}}_{90^\circ\ (\frac{\pi}{2})\text{ phase lead of position}}$$

$$\ddot{x}(t) = -\omega^2\bar{x}e^{i\omega t} \leftarrow -1 = \cos\pi + i\sin\pi = e^{i\pi} \rightarrow \underbrace{\ddot{x}(t) = \omega^2\bar{x}e^{i(\omega t + \pi)}}_{180^\circ\ (\pi)\text{ phase lead of position}}$$