**Step Inputs (Sec. 2.2)**

Instantaneous change between 2 (desired) states/setpoints

Can be a change in position, velocity, accel., force, etc.

ex) Move from point A to point B right now  
    Change velocity from 0 to 60 mph (right now)

Note: The book calls "position-based" inputs seismic inputs.

Intuitively, we know that the solution is oscillation about the steady-state offset, so let's assume a solution of the form:

\[ x(t) = x_d + e^{\gamma t} (a \cos \omega t + b \sin \omega t) \]

\[ x_d \quad \text{Steady-state offset} \]
\[ (a \cos \omega t + b \sin \omega t) \quad \text{Vibration around it} \]

Note: The book calls this variable \( c \)

Now, just use initial conditions to solve for \( a \) and \( b \):

\[ x(0) = 0 \]
\[ x(0) = x_d + e^{\gamma 0} (a \cos \omega 0 + b \sin \omega 0) = x_d \]

\[ \alpha = -x_d \]

at \( t = 0^+ \) (instant after the step), the system is \(-x_d \) below its new equilibrium.

\[ x(0^+) = -x_d \]

\[ x(0) = x_d + e^{\gamma 0} (a \cos \omega 0 + b \sin \omega 0) + e^{\gamma 0} (-\alpha \sin \omega 0 + \omega \beta \cos \omega 0) = 0 \]

\[ x(0) = -x_d + \omega \beta \]

\[ b = \frac{\omega \beta}{\omega \beta} = -\frac{-x_d}{\omega \beta} = \frac{-x_d}{\omega \beta} \]

\[ \omega = \sqrt{\frac{k}{m}} \]

\[ \beta = \sqrt{\frac{c}{m}} \]
So, the step response (to a position input) is:

\[ x(t) = x_0 + e^{-\frac{t}{\tau}} \left( x_0 \cos(\omega_n t) - \frac{x_0}{\tau} \sin(\omega_n t) \right) \]

Setpoint (often the input) vibration around it

\[ \tau_r = \text{rise time} - \text{time between } 10\% \text{ and } 90\% \text{ of steady-state value.} \]
\[ T_s = \text{settling time} - \text{time for oscillation to die to } x_9\% \text{ of the steady-state} \]
\[ \eta = \text{overshoot} - \text{amount by which the response peak exceeds the steady-state} \]
\[ (x_9\% \text{ is often } 2\%) \]

Controls Preview/Review

For 2nd-order underdamped systems, all of these performance measures can be written in closed-form as functions of the natural frequency and damping ratio.

They are also often used as specifications for control systems.
Harmonic, Seismic (Position) Inputs (Sec. 2.3)

Now, we want to continue to move the mass "back-and-forth" indefinitely.

Q: What happens to this system response \( x \) if we *slowly* move \( y \) (the input) between two positions?

It should track \( y \) pretty well.

Q: What if we change our input \( y \) rapidly?

(Against) nothing. We create very little motion in \( x \).

Q: What if we move \( y \) "back-and-forth" at the system's natural frequency?

Theory - amp. of vibration increases toward infinity \( <--- \) Resonance

Actual - amp. increases until something breaks.
$$m x = F_p$$

$$m x = k(y - x) \rightarrow m x + k x = k y \rightarrow x + \frac{k}{m} x = \frac{1}{k} y$$

$$x + \omega_n^2 x = \omega_n^2 y$$

Let's model the harmonic excitation as a sine wave

$$y(t) = A \sin(\omega t) \quad \omega = \text{freq of input and } A = \text{amplitude}$$

If $\omega = \omega_n$, we can assume a solution for $x(t)$ that has the form:

$$x(t) = A \sin(\omega_n t + \phi)$$

Note: $w \neq w_n$, $w$ is the input frequency

Note: This is the steady-state solution. The full solution would have some transient vibration.

Now, substitute this assumed solution into the equation of motion

$$x(t) = \omega_n^2 \sin(\omega_n t + \phi) \quad \text{and} \quad x(t) = -\omega_n^2 \sin(\omega_n t + \phi)$$

$$\frac{d^2}{dx^2} x + \omega_n^2 x = \omega_n^2 y$$

$$\left[-\omega_n^4 \sin(\omega_n t + \phi)\right] + \omega_n^2 \left[A \sin(\omega_n t + \phi)\right] = \omega_n^2 y \sin(\omega_n t + \phi) \quad \text{--- new called the sin and cos terms}$$

$$(-\omega_n^2 \cdot \omega_n^2) \sin(\omega_n t + \phi) + (-\omega_n^2 \cdot \omega_n^2) \cos(\omega_n t + \phi) = \omega_n^2 y \sin(\omega_n t + \phi)$$

For this solution to hold, both sin and cos terms must match, so:

$$\frac{\text{sin terms}}{\text{cos terms}}$$

$$-\omega_n^2 + \omega_n^2 = 0$$

$$b\omega_n^2 - b\omega_n^2 = 0 \rightarrow b = 0 \quad \text{--- So, cos part of the solution disappears}$$

(Remember this is the undamped solution)
Undamped Response to Harmonic, Seismic Inputs (cont.)

**Q:** What does the cosine terms disappearing mean physically? (given the input is a sine wave)

The response is either exactly:

*in-phase* - x is always moving in the same direction as y

or

*out-of-phase* - x is always moving in the opposite direction of y

### Solving for the sine components

\[-\omega_0^2 + \omega^2 = \omega_n^2 \]

\[ \Rightarrow \ \alpha = \frac{\omega_n^2}{\omega_0^2 - \omega^2} \]

\[ x(t) = \frac{\omega_n^2}{\omega_0^2 - \omega^2} \sin \omega t \quad \text{or (more generally)} \quad x(t) = \left| \frac{\omega_n^2}{\omega_0^2 - \omega^2} \right| \sin (\omega t + \phi) \]

\( \phi \) = phase shift of the rep from the input

**Q:** When is the response exactly in-phase?

At input frequencies lower than the natural frequency (\( \omega < \omega_n \))

**Q:** When is the response exactly out-of-phase?

At input frequencies higher than the natural frequency (\( \omega > \omega_n \))

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![Amplitude (includes phase)](image1)

![Magnitude (no phase information)](image2)
Normalization

We like to have more "general" ideas of the solution. So, let's normalize to remove the natural frequency from the response.

To do so, let's divide the numerator and denominator by the natural frequency squared.

\[ x(t) = \left[ \frac{\omega_n^2}{\omega_n^2 - \omega^2} \right] \tilde{y} \sin(\omega t) \]

Let's plot this.

\[ \bar{x} = \frac{x(t)}{\omega_n^2} \]

Define \( \bar{x} = \frac{x_n^2}{\omega_n^2} \)

\( \bar{x} = \frac{\omega_n^2 / \omega_n^2 - \omega_n^2}{1 - \omega_n^2} = \frac{1}{1 - \omega_n^2} \), where \( \omega = \frac{\omega}{\omega_n} \)

Normalized, non-dimensional freq

So, \( x(t) = \frac{1}{1 - \omega_n^2} \tilde{y} \sin(\omega t) = \frac{1}{1 - \omega_n^2} \tilde{y}(t) \)

Look at this term. It's just scaling the input.

Amplitude (includes phase)

Magnitude (no phase information)

Normalized Frequency \( \left( \Omega = \frac{\omega}{\omega_n} \right) \)
Q: What would a similar (frequency response) plot for the acceleration of \( m \) look like?

\[
x(t) = \frac{-\omega_n^2}{\omega^2 - \omega_n^2} \tilde{y} \sin \omega t
\]

So

\[
x(t) = \frac{-\omega_n^2 \omega^2}{\omega_n^2 - \omega^2} \tilde{y} \sin \omega t
\]

Input Frequency \((\omega)\)

Q: When might we care about acceleration?

When a human involved – we "feel" acceleration \((F=ma)\)
(or force/accel. sensitive equipment)

Q: What happens if the input is a "direct" force instead?

We should expect similar characteristics.
Direct Force Excitation with No Damping (Sec. 2.4)

\[ mx = -F_x + f = -kx + f \]

\[ m\ddot{x} + kx = f \quad \text{or} \quad (\text{directly}) \quad \ddot{x} + \frac{k}{m} x = \frac{1}{m} f \]

Assume \( f(t) = \bar{f}\cos\omega t \Rightarrow \text{expect the solution to be of form} \quad x(t) = \bar{x}\cos\omega t \)

Now, put this assumed solution into the eq of motion and solve for \( \bar{x} \)

\[ (-\omega^2 \bar{x}\cos\omega t) + \omega^2 (\bar{x}\cos\omega t) = \frac{1}{m}\bar{f}\cos\omega t \]

\[ \omega_n^2 - \omega^2 = \frac{1}{m} \bar{f} \]

\[ \bar{x} = \frac{\bar{f}}{m\left(\omega_n^2 - \omega^2\right)} \quad \text{This is the amplitude of vibration of } m \]

So, \( x(t) = \frac{1}{m(\omega_n^2 - \omega^2)} \bar{f}\cos\omega t = \left[\frac{1}{m(\omega_n^2 - \omega^2)}\right] f(t) \quad \text{Let's plot this term} \)

Amplitude (includes phase)

\[
\frac{1}{m(\omega_n^2 - \omega^2)}
\]

Input Frequency \( (\omega) \)

Magnitude (no phase information)

\[
\frac{1}{\sqrt{\frac{\omega_n^2 - \omega^2}{m^2}}}
\]

Input Frequency \( (\omega) \)
Normalization

We can also normalize this by using the nondimensional frequency, $\Omega$. To even further normalize, we can scale the amplitude such that it is 1 when the excitation frequency is 0.

$$
\chi(t) = \frac{1}{m\omega_0^2} \tilde{F} \sin \omega_0 t \rightarrow \frac{1}{m\left(\frac{\omega_0}{\omega_0^2} - \frac{\omega^2}{\omega_0^2}\right)} \tilde{F} \sin \omega_0 t = \frac{1}{m\omega_0^2 (1 - \Omega^2)} \tilde{F} \sin \omega_0 t
$$

$$
\frac{m \omega_0^2}{\tilde{F}} \chi \text{ vs. } \Omega
$$

Amplitude (includes phase)

Magnitude (no phase information)