Chapter 2 – Single DOF Forced Vibration

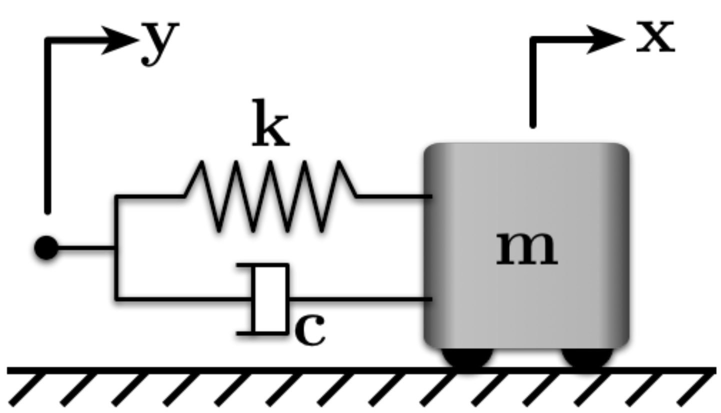
Step Inputs (Sec. 2.2)

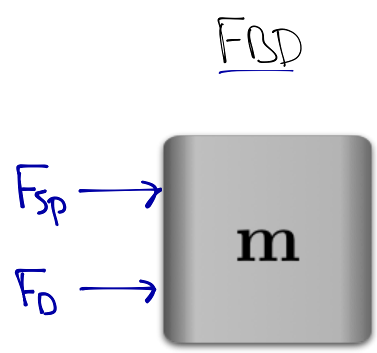
Instantaneous change between 2 (desired) states/setpoints

Can be a change in position, velocity, accel., force, etc.

ex) Move from point A to point B right now
Change velocity from 0 to 60 mph (right now)

Note: The book calls "position-based" inputs seismic inputs.





$$m\ddot{x} = -F_{sp} - F_D$$
$$= k(y-x) + c(\dot{y}-\dot{x})$$
$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 = 2\zeta\omega_n\dot{y} + \omega_n^2 y$$

Intuitively, we know that the solution is oscillation about the steady-state offset, so let's assume a solution of the form:

$$x(t) = \underbrace{x_d}_{\text{Steady-state offset}} + \underbrace{e^{-\zeta\omega_n t}(a\cos\omega_d t + b\sin\omega_d t)}_{\text{Vibration around it}}$$

Note: The book calls this variable c

Now, just use initial conditions to solve for a and b:

$x(0) = 0$

$x(0) = x_d + a = 0$

$$\underbrace{a = -x_d}$$

at $t = 0^+$ (instant after the step), the system is $-x_d$ "below" its new equilibrium.

$\dot{x}(0) = 0$

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(a\cos\omega_d t + b\sin\omega_d t) + e^{-\zeta\omega_n t}(-a\omega_d\sin\omega_d t + b\omega_d\cos\omega_d t)$$

$$\dot{x}(0) = -\zeta\omega_n a + b\omega_d = 0$$

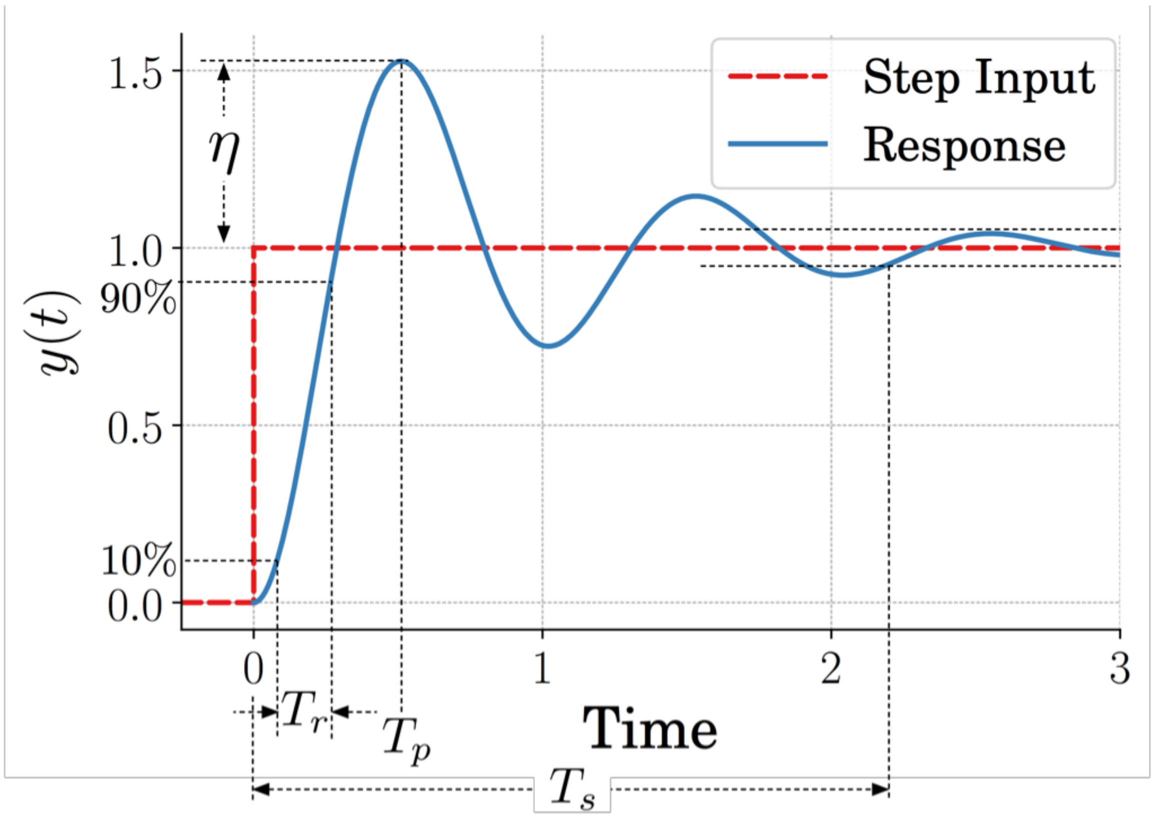
$$b = \frac{\zeta\omega_n a}{\omega_d} = \frac{-\zeta\omega_n x_d}{\omega_n\sqrt{1-\zeta^2}} = \frac{-\zeta x_d}{\sqrt{1-\zeta^2}}$$

Step Inputs (cont.)

So, the step response (to a position input) is:

$$x(t) = \underset{\substack{\uparrow \\ \text{setpoint} \\ \text{(often the "input")}}}{x_d} + \underbrace{e^{-\zeta\omega_n t}\left(-x_d \cos\omega_d t - \frac{\zeta x_d}{\sqrt{1-\zeta^2}} \sin\omega_d t\right)}_{\text{vibration around it}}$$

Remember that $x_d = c$ to match book derivation



T_r = rise time – time between 10% and 90% of steady-state value

T_s = settling time – time for oscillation to die to X% of the steady-state (X% is often 2%)

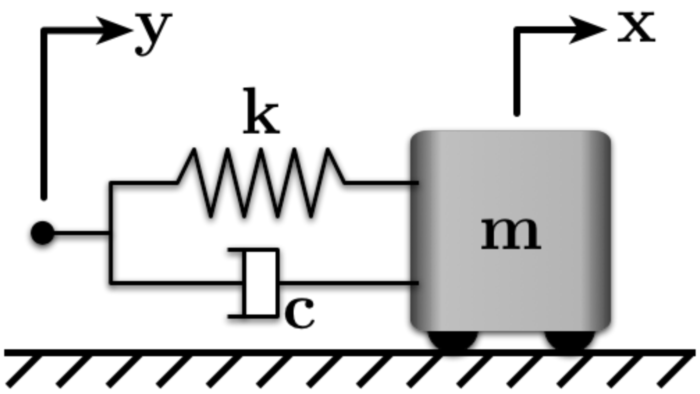
η = overshoot – amount by which the response peak exceeds the steady-state (often measured as a % of the steady-state)

Controls Preview/Review

For 2nd-order underdamped systems, all of these performance measures can be written in closed-form as functions of the natural frequency and damping ratio.

They are also often used as specifications for control systems.

Harmonic, Seismic (Position) Inputs (Sec. 2.3)



Now, we want to continue to move the mass "back-and-forth" indefinitely.

Q: What happens to this system respose (x) if we *slowly* move y (the input) between two positions?

It should track y pretty well.

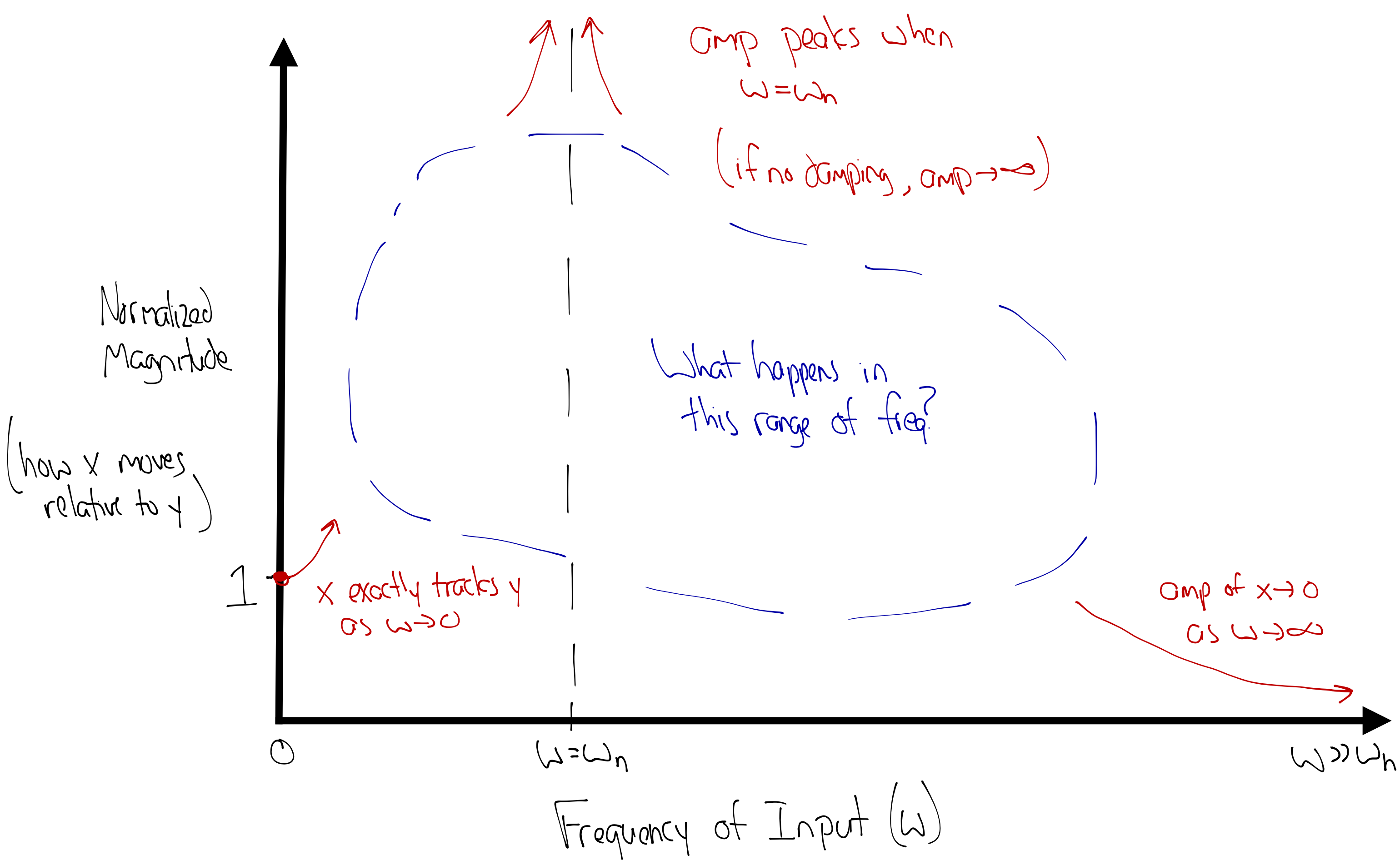
Q: What if we change our input (y) rapidly?

(Almost) nothing. We create very little motion in x.

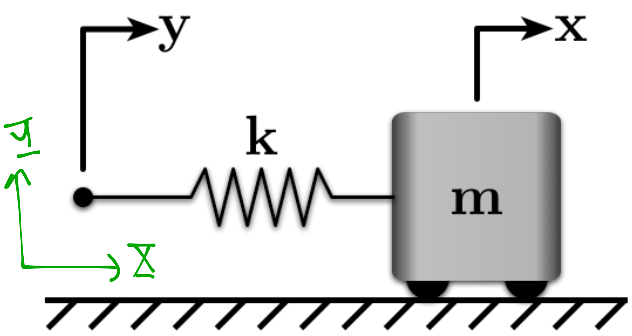
Q: What if we move y "back-and-forth" at the system's natural frequency?

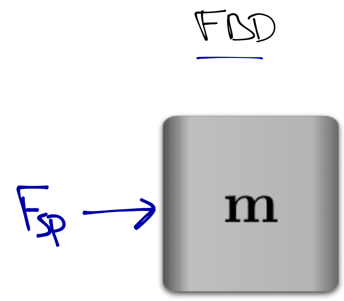
Theory - amp. of vibration increases toward infinity <--- Resonance

Actual - amp. increases until somthing breaks.



Undamped Response to Harmonic, Seismic Inputs





Controls Preview/Review
This is identical to a mass under proportional control.

$m\ddot{x} = F_{sp}$

$m\ddot{x} = k(y-x) \rightarrow m\ddot{x} + kx = ky \rightarrow (\text{divide by } m) \rightarrow \ddot{x} + \frac{k}{m}x = \frac{k}{m}y$

$\ddot{x} + \omega_n^2 x = \omega_n^2 y$

Let's model the harmonic excitation as a sine wave

$y(t) = \bar{y}\sin(\omega t)$ $\qquad$ ω = freq of input and $\bar{y}$ is amplitude

If $\omega \neq \omega_n$, we can assume a solution for $x(t)$ that has the form:

$x(t) = a\sin\omega t + b\cos\omega t$ $\leftarrow$ Note: $\omega \neq \omega_n$, ω is the input frequency

Note: This is the steady-state solution. The full solution would have some transient vibration.

Now, substitute this assumed solution into the equations of motion:

$\dot{x}(t) = a\omega\cos\omega t - b\omega\sin\omega t$ $\qquad$ and $\qquad$ $\ddot{x}(t) = -a\omega^2\sin\omega t - b\omega^2\cos\omega t$

$\ddot{x} + \omega_n^2 x = \omega_n^2 y$

$[-a\omega^2\sin\omega t - b\omega^2\cos\omega t] + \omega_n^2[a\sin\omega t + b\cos\omega t] = \omega_n^2\bar{y}\sin\omega t$ — now collect the sin and cos terms

$(-a\omega^2 + a\omega_n^2)\sin\omega t + (-b\omega^2 + b\omega_n^2)\cos\omega t = \omega_n^2\bar{y}\sin\omega t$

For this solution to hold, both sin and cos terms must match, so:

sin-terms

$-a\omega^2 + a\omega_n^2 = \omega_n^2\bar{y}$

cos terms

$b\omega_n^2 - b\omega^2 = 0 \rightarrow b = 0$ $\leftarrow$ So, cos part of the solution disappears (Remember: This is the undamped solution.)

Undamped Response to Harmonic, Seismic Inputs (cont.)

Q: What does the cosine terms disappearing mean physically?
(given the input is a sine wave)

The response is either exactly:

in-phase - x is always moving in the same direction as y

or

out-of-phase - x is always moving in the opposite direction of y

Solving for the sine components

$$-a\omega^2 + a\omega_n^2 = \omega_n^2 \bar{y} \longrightarrow a = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y} \quad \text{so,}$$

$$x(t) = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\bar{y}\sin\omega t \quad \text{or (more generally)} \quad x(t) = \left|\frac{\omega_n^2}{\omega_n^2 - \omega^2}\right|\bar{y}\sin(\omega t + \phi)$$

$\phi \equiv$ phase shift of the resp from the input

Q: When is the response exactly in-phase?

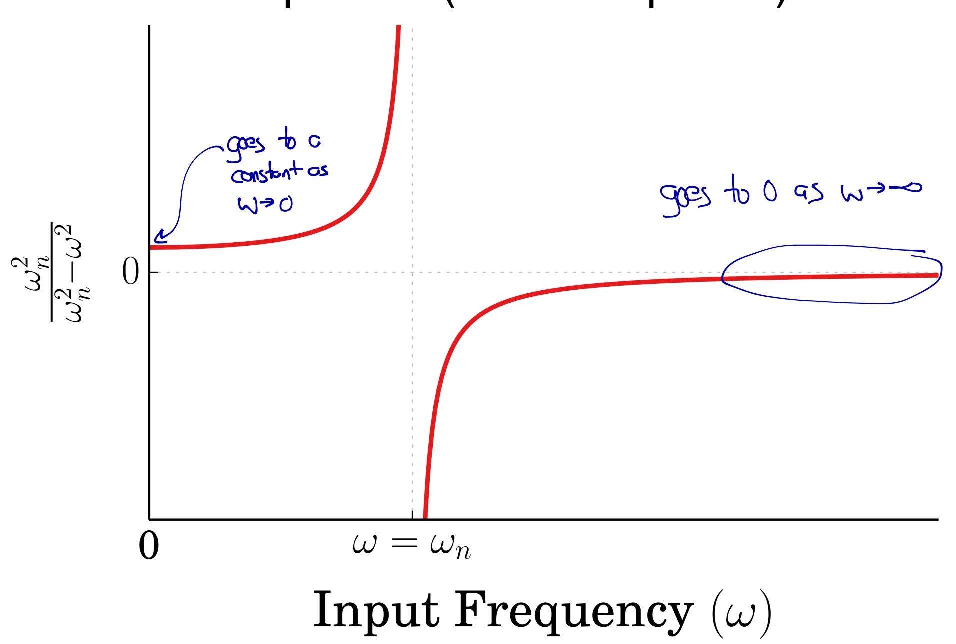
At input frequencies lower than the natural frequency ($\omega < \omega_n$)

Q: When is the response exactly exactly out-of-phase?

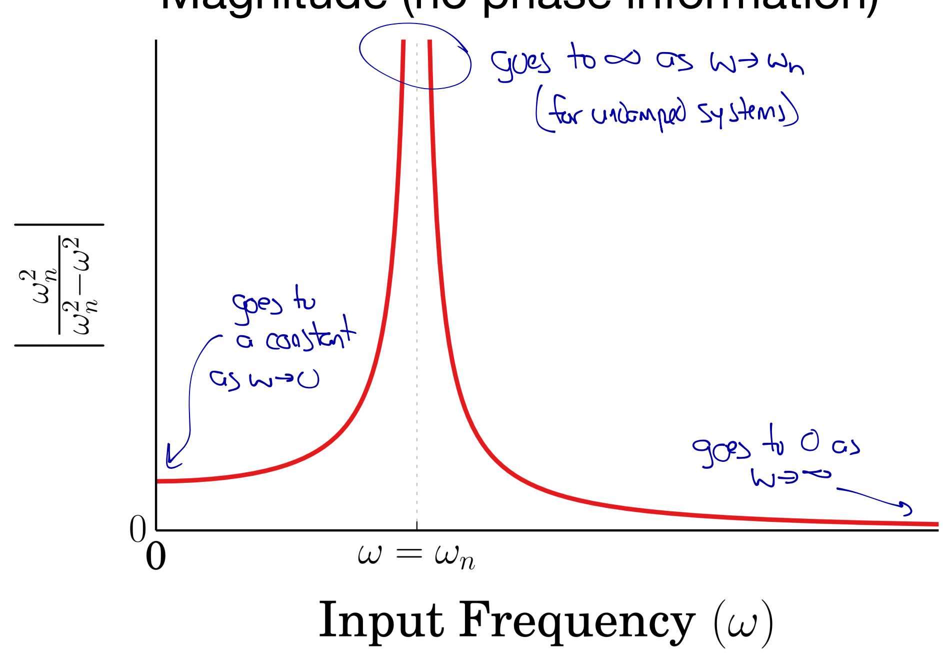
At input frequencies higher than the natural frequency ($\omega > \omega_n$)

Look at the sign of $\frac{\omega_n^2}{\omega^2 - \omega_n^2}$

Amplitude (includes phase)



Magnitude (no phase information)



Normalization

We like to have more "general" ideas of the solution. So, let's normalize to remove the natural frequency from the response.

To do so, let's divide the numerator and denominator by the natural frequency squared.

We found the solution to be $x(t) = \left[\frac{\omega_n^2}{\omega_n^2-\omega^2}\right]\left[\bar{Y}\sin\omega t\right]$ ← $Y(t) = \bar{Y}\sin\omega t$, remember?

↑ Look at this term. It's just scaling the input

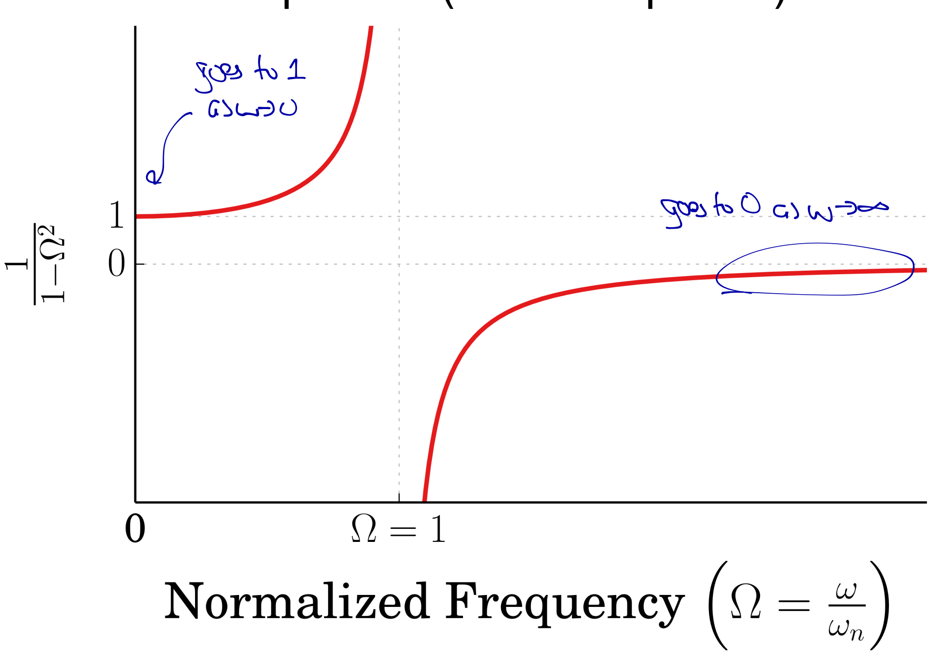
Define $\bar{X} = \frac{\omega_n^2}{\omega_n^2-\omega^2}$ ← Now divide num. and den. by ω_n^2

$$\bar{X} = \frac{\omega_n^2/\omega_n^2}{\omega_n^2/\omega_n^2 - \omega^2/\omega_n^2} = \frac{1}{1-\Omega^2}$$

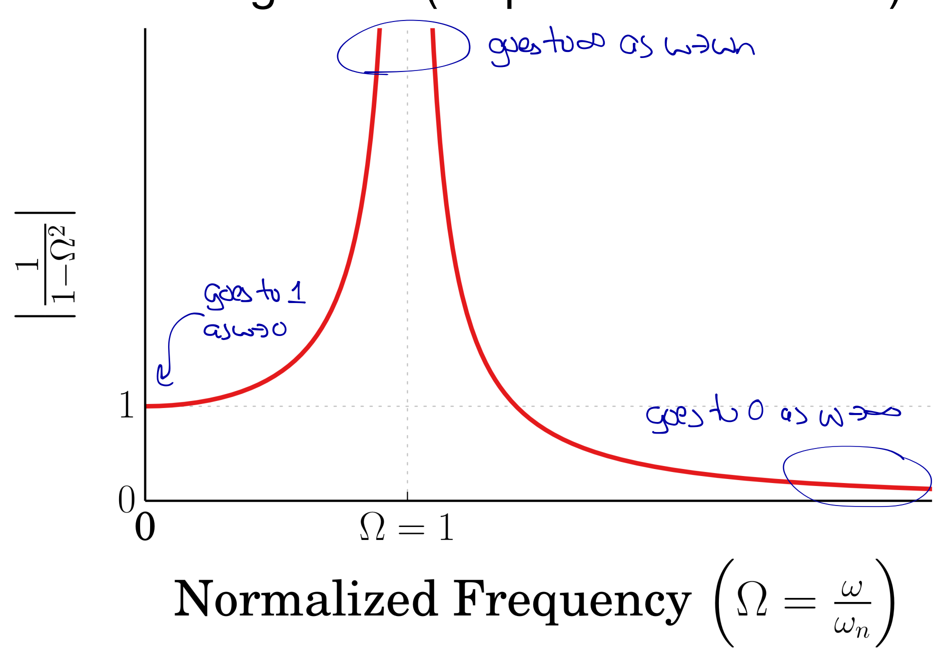
, where $\Omega = \frac{\omega}{\omega_n}$ } Normalized, non-dimensional freq.

$$\text{So, } x(t) = \frac{1}{1-\Omega^2}\bar{Y}\sin\omega t = \underbrace{\frac{1}{1-\Omega^2}}_{\text{Let's plot this.}} Y(t)$$

Amplitude (includes phase)

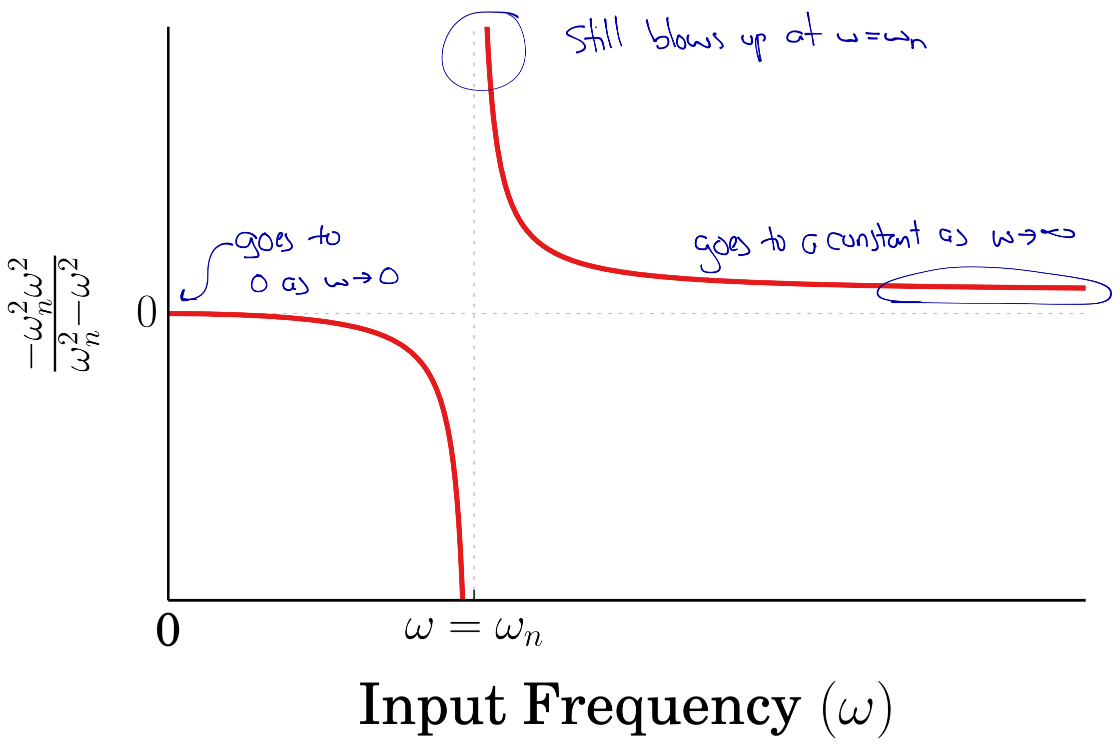


Magnitude (no phase information)



Q: What would a similar (frequency response) plot for the acceleration of m look like?

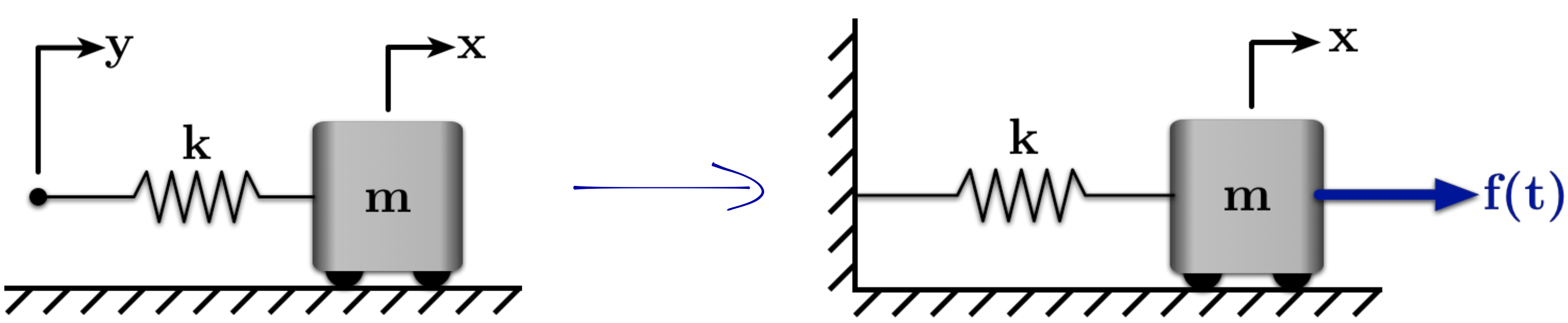
$$x(t) = \frac{\omega_n^2}{\omega_n^2 - \omega^2}\,\bar{Y}\sin\omega t \qquad \text{so} \qquad \ddot{x}(t) = \frac{-\omega_n^2\omega^2}{\omega_n^2 - \omega^2}\,\bar{Y}\sin\omega t$$



Q: When might we care about acceleration?

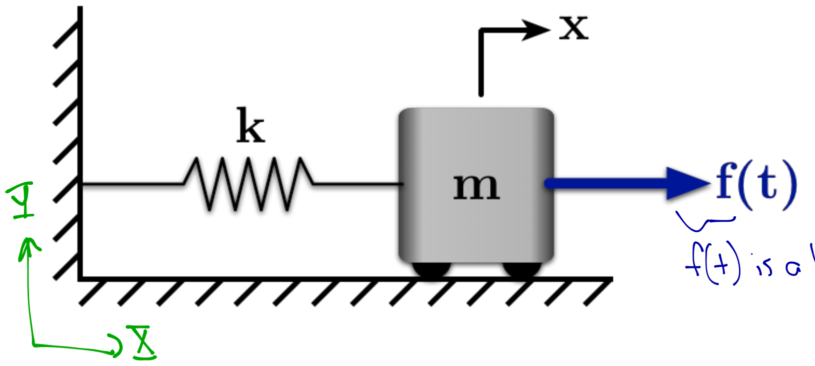
When a human involved – we "feel" accleration (F=ma)
(or force/accel. sensitive equipment)

Q: What happens if the input is a "direct" force instead?



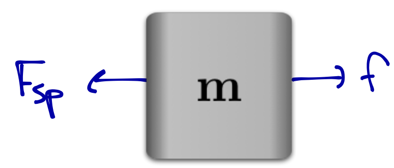
We should expect similar characteristics.

Direct Force Excitation with No Damping (Sec. 2.4)



f(t) is a harmonic force

FBD



$$m\ddot{x} = -F_{sp} + f = -kx + f$$

$$m\ddot{x} + kx = f \quad \underline{\text{or}} \quad (\text{divide by } m) \quad \ddot{x} + \omega_n^2 x = \frac{1}{m} f$$

Assume $f(t) = \bar{f} \sin\omega t \longrightarrow$ expect the solution to be of form $x(t) = \bar{X} \sin\omega t$

Now, just plug this assumed solution into the eq. of motion and solve for $\bar{X}$

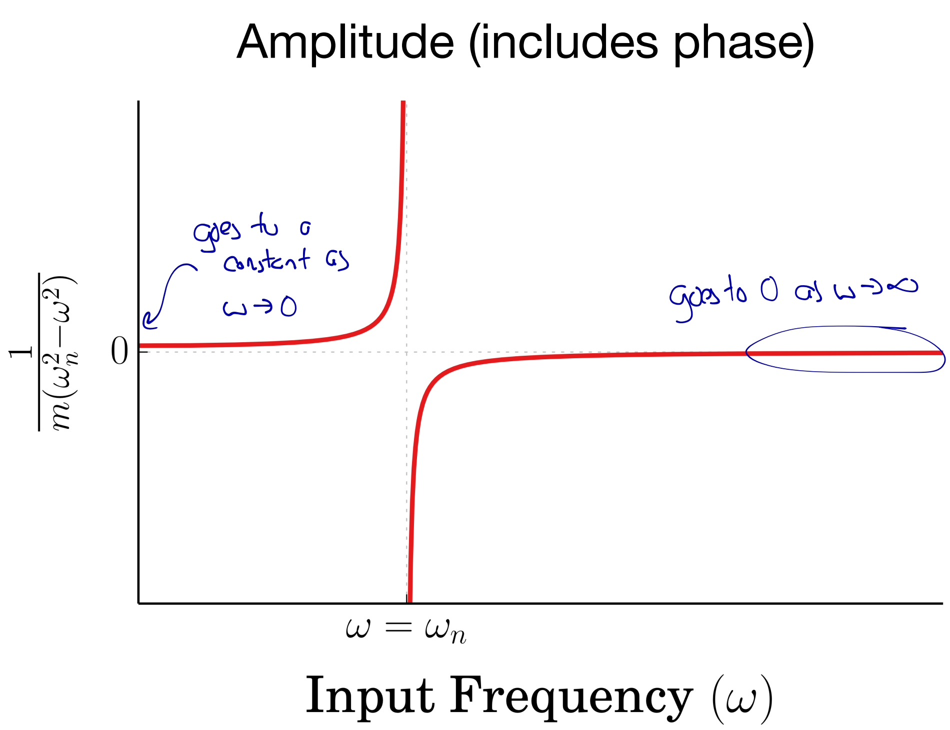
$$\left(-\omega^2 \bar{X} \cancel{\sin\omega t}\right) + \omega_n^2\left(\bar{X} \cancel{\sin\omega t}\right) = \frac{1}{m} \bar{f} \cancel{\sin\omega t}$$

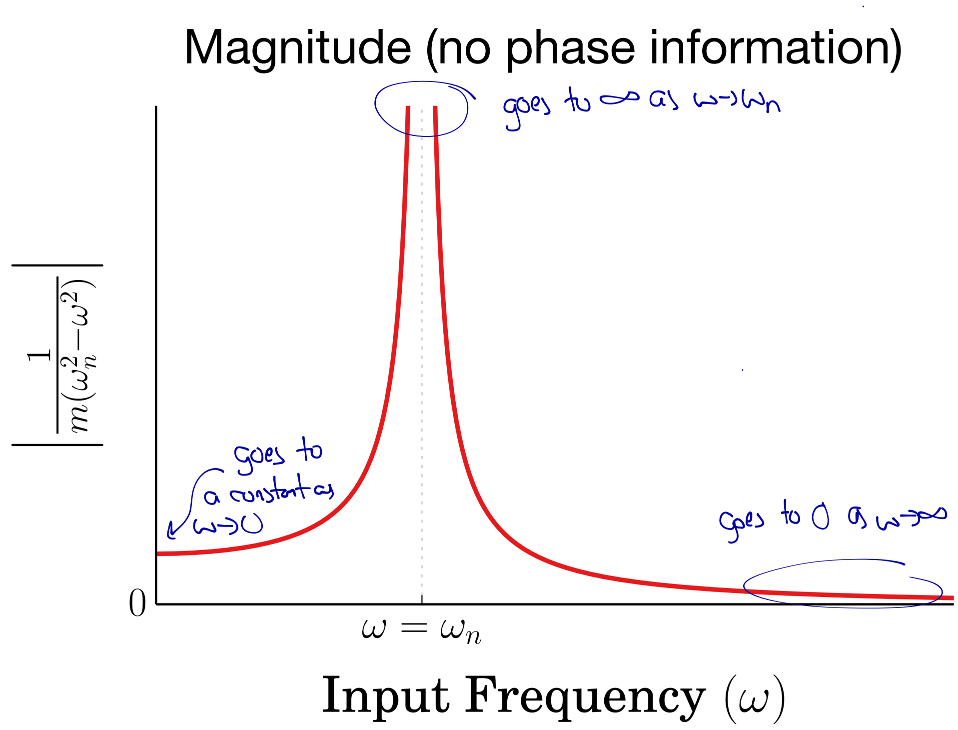
$$\left(\omega_n^2 - \omega^2\right) \bar{X} = \frac{1}{m} \bar{f}$$

$$\bar{X} = \frac{\bar{f}}{m} \left(\frac{1}{\omega_n^2 - \omega^2}\right) \longleftarrow \text{This is the amplitude of vibration of } m$$

$$\text{So,} \quad x(t) = \frac{1}{m(\omega_n^2 - \omega^2)} \bar{f} \sin\omega t = \underbrace{\left[\frac{1}{m(\omega_n^2 - \omega^2)}\right]} f(t)$$

Let's plot this term.





Normalization

We can also normalize this by using the nondimensional frequency, Ω .

To even further normalize, we can scale the amplitude such that it is 1 when the excitation frequency is 0.

$$x(t) = \frac{1}{m(\omega_n^2-\omega^2)}\bar{f}\sin\omega t \longrightarrow \frac{1/\omega_n^2}{m\left(\frac{\omega_n^2}{\omega_n^2}-\frac{\omega^2}{\omega_n^2}\right)}\bar{f}\sin\omega t = \underbrace{\frac{1}{m\omega_n^2(1-\Omega^2)}\bar{f}}_{\bar{x}=\frac{\bar{f}}{m\omega_n^2(1-\Omega^2)}}\sin\omega t$$

Plot $\frac{m\omega_n^2}{\bar{f}}\bar{x}$ vs. Ω