A Less-Simple Example (Newton/Euler)

Q: What is $F_{sp}$?

Look at the horizontal displacement (assuming small angle) of the connection point

$X = a \sin \theta$

If $\theta = 0$ is the equal, then this is the $\delta$ of the spring

$F_{sp} = k \delta = ka \sin \theta$

Q: Where should we sum moments about?

Point O - pure rotation about O ($x,y$) and $F_{eq}: 0$ so reactions don't contribute

$I_{o} \ddot{\theta} = \sum \vec{M}_o = (\vec{F}_{rp} \times \vec{F}_{rp}) + (\vec{F}_{mb} \times \vec{F}_{sp})$

$= \left[ (-a \sin \theta \vec{i} + a \cos \theta \vec{j}) \times (ka \sin \theta \vec{i}) \right] + \left[ (l \sin \theta \vec{i} + l \cos \theta \vec{j}) \times (-mg \vec{j}) \right]$}

$I_{o} \ddot{\theta} = \left[ -ka^2 \cos \theta \sin \theta \vec{i} + \text{mg} l \sin \theta \vec{j} \right]$}

$I_{o} \ddot{\theta} + ka^2 \cos \theta \sin \theta - \text{mg} l \sin \theta = 0$ ← Nonlinear

Q: How can we linearize?

Assume small angles, $\sin \theta \approx \theta$ and $\cos \theta = 1$ ← Also assume $\theta_m = 0$

$I_o \ddot{\theta} + (ka^2 - \text{mg}) \theta = 0$ → $I_o = ml^2$ → $m \ddot{\theta} + (k \theta - \text{mg}) \theta = 0$ → $\ddot{\theta} + \left(\frac{k}{ml^2} \text{mg}\right) \theta = 0$ ← Linearized

Lack of inertia

Q: What happens if the spring is too weak?

It falls over... the system is unstable → $\theta_m \neq 0$ stable

Q: What is the critical value of $k$? Need $k \theta > \text{mg}$ → $k > \frac{\text{mg}}{\theta_m}$
A Less-Simple Example (Lagrange)

Q: What should we choose as the generalized coord. — θ

The system is in pure rotation about point O, so

\[ T = \frac{1}{2} I_0 \dot{\theta}^2 \]

The is potential energy from gravity and the spring

Set the gravity datum to the pin location → \( V_{gr} = mg \ell \cos \theta \)

The spring potential is \( \frac{1}{2} k \ell^2 \). If we assume small angles \( \ell \approx x \), so \( V_p = \frac{1}{2} kx^2 \)

But, we need to write all energy in term of the generalized coord., so \( x \cos \theta \rightarrow \ell \approx \frac{1}{2} k(\ell \sin \theta)^2 \)

\[ V = \frac{1}{2} k\ell^2 \sin^2 \theta + mg \ell \cos \theta \]

\[ L = T - V = \frac{1}{2} I_0 \dot{\theta}^2 - \left( \frac{1}{2} k\ell^2 \sin^2 \theta + mg \ell \cos \theta \right) \]

Now, plug into Lagrange's Equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \]

\[ \frac{d}{dt} \left( I_0 \dot{\theta} \right) - I_0 \ddot{\theta} = - \left( k\ell^2 \cos \theta \sin \theta - mg \ell \sin \theta \right) \]

\[ I_0 \ddot{\theta} + k\ell^2 \cos \theta \sin \theta - mg \ell \sin \theta = 0 \]

← Some as we got with Newton-Euler (It better be!)

From here, we could linearize as before.
Lagrange's Method with Viscous Damping

Remember that \( F_D = C \dot{d} \) where \( \dot{d} \) is the rate of change of the length of the damper.

To match the form needed for Lagrange (remember, it's energy-based), integrate with respect to \( d \):

\[
\int F_D \, dd = \int (C \dot{d}) \, dd = \frac{1}{2} C \dot{d}^2 \quad \text{Looks similar to } V_p, \text{ right?}
\]

Rayleigh's Dissipation Function (RD) as

\[
RD = \frac{1}{2} C \dot{d}^2 \quad \text{Note: Only valid for linear, viscous dampers}
\]

Use this to represent the damper in Lagrange's Equations:

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial RD}{\partial q_i} - \frac{\partial L}{\partial q_i} = 0 \quad \text{for } i = 1, \ldots, n \quad \text{Note: Still no external forces.}
\]
Lagrange's Method with External Forces

To understand external forces, we need to introduce...

Virtual Displacements

* Infinitesimally small changes in generalized coordinates
* Occur in zero time (no time elapses during the move)
* Do not violate system constraints

Consider $f(q_1, q_2, ..., t) = 0$ as a system constraint. If we allow virtual displacements $\delta q_1, \delta q_2, ...$ then

$$f(q_1, \delta q_1, q_2, \delta q_2, ..., t) = 0$$

No time has passed

Why does this help? → It gives us a way to determine virtual work

Why does that help? → It tells us how to determine the contribution of each external force in the direction of each generalized coord

Principle of Virtual Work

$$\delta W = \sum \bar{F}_j \cdot \delta \vec{r}_j + \sum \bar{M}_n \cdot \delta \theta_n$$

$\delta \vec{r}_j$ = virtual displacement of point of application of $\bar{F}_j$

$\delta \theta_n$ = virtual rotation of body on which $\bar{M}_n$ acts

And

$$\delta W_i = Q_i \delta q_i$$

$Q_i$ = component of the external force acting in direction $\delta q_i$ → component of force that causes motion in $q_i$

Now, we can write the "total" form of Lagrange's Equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial R}{\partial q_i} - \frac{\partial L}{\partial q_i} = Q_i \quad \text{for } i = 1, 2, ..., n$$
An Even-Less-Simple Example (Ex. 1.14-1.15)

2 bodies, 2 DoF \( \rightarrow \) 2 generalized co"ons and 2 Eq. of motion

Use generalized co"ons \((X, \theta)\)

The velocity of \(m_1\) is \(x\)

Q: What is the velocity of \(m_2\)?

It depends on both \(x\) and \(\theta\)

\[
\vec{v}_2 = \dot{x} \hat{i} + l \dot{\theta} \hat{b}_\theta
\]

\[
\vec{v}_a = \cos \theta \hat{i} + \sin \theta \hat{j}
\]

\[
\vec{v}_2 = (x \dot{\theta} \cos \theta) \hat{i} + (l \dot{\theta} \sin \theta) \hat{j}
\]

Q: What should we choose as the gravity datum?

Let's choose the attachment point

(The book chooses the "lowest" pendulum point)

Now, form the energies, then the Lagrangian

\[
T = \frac{1}{2} m_1 x^2 + \frac{1}{2} m \dot{y}_2 \dot{y}_2
\]

\[
= \frac{1}{2} m_1 x^2 + \frac{1}{2} m (x + l \dot{\theta} \cos \theta)^2 + (l \dot{\theta} \sin \theta)^2
\]

\[
T = \frac{1}{2} m_1 x^2 + \frac{1}{2} m [(x + l \dot{\theta} \cos \theta)^2 + (l \dot{\theta} \sin \theta)^2]
\]

\[
\text{Total kinetic energy}
\]

\[
V = V_{gc} + V_{sp}
\]

\[
V_{gc} = mg \dot{y} = -mg \dot{\theta} \cos \theta
\]

\[
V_{sp} = \frac{1}{2} k x^2
\]

\[
V = -mg \dot{\theta} \cos \theta + \frac{1}{2} k x^2
\]

\[
\text{Total potential energy}
\]

\[
L = T - V
\]

\[
\text{Lagrangian}
\]

Q: What about the external force, \(f(t)\)?

We have to find the \(Q_i\) terms for Lagrange's EQ.

To do so we'll use virtual displacements and virtual work.

We know that

\[
\vec{v}_2 = (x + l \dot{\theta} \cos \theta) \hat{i} + (l \dot{\theta} \sin \theta) \hat{j}
\]

This is the point the force is applied with

\[
\frac{d \vec{r}_2}{dt} = (\dot{x} + l \ddot{\theta} \cos \theta) \hat{i} + (l \ddot{\theta} \sin \theta) \hat{j}
\]

\[
\text{virtual displacement of } m
\]
An Even-Less-Simple Example (cont.)

To find \( Q_i \), we need to find the virtual work done by the force \( (\text{Force} \cdot \text{Virtual Displacement}) \)

\[
\int W = \int \mathbf{F} \cdot \mathbf{d} \mathbf{r} = \mathbf{F} \cdot \left[ \left( \mathbf{d} x \times l \theta \cos \theta \right) \mathbf{i} + \left( l \theta \sin \theta \right) \mathbf{j} \right] = F \left( \mathbf{d} x \times l \theta \cos \theta \right)
\]

\[
\int W = F \mathbf{d} x + (F \mathbf{cos \theta}) \mathbf{d} \theta
\]

Now, apply Lagrange's Eq for each \( q_k \) - \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k
\]

\[
Q_1 = x
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial x} \right) - \frac{\partial L}{\partial x} = Q_1
\]

\[
\frac{\partial L}{\partial x} = m_1 x + \frac{1}{2} m \left[ 2(x + l \theta \cos \theta) \right] \quad \frac{\partial L}{\partial \theta} = -k x
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \theta} \right) = m_1 x + m \left( x + l \dot{\theta} \cos \theta - l \theta \sin \theta \right) \quad Q_1 = F
\]

\[
(m_1 + m) x + ml^2 \dot{\theta} \cos \theta - ml^2 \dot{\theta} \sin \theta + kx - F
\]

\[
Q_2 = 0
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = Q_2
\]

\[
\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \left[ 2(x + l \theta \cos \theta)(l \cos \theta) + 2(l \theta \sin \theta)(l \sin \theta) \right]
\]

\[
= m \left[ l x \cos \theta + l^2 \dot{\theta} \cos \theta + l \theta \sin \theta \right] = m(l x \cos \theta + l^2 \dot{\theta})
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = m(l x \cos \theta - l x \sin \theta + l \dot{\theta})
\]

\[
\frac{\partial L}{\partial \theta} = \frac{1}{2} m \left[ 2(x + l \theta \cos \theta)(l \sin \theta) + 2(l \theta \sin \theta)(l \theta \cos \theta) \right] - mg l \sin \theta
\]

\[
= m [-l x \dot{\theta} \sin \theta + l \dot{\theta} \sin \theta]
\]

\[
Q_2 = F l \cos \theta
\]

\[
m(l x \cos \theta - l x \sin \theta + l \dot{\theta}) - m(l x \sin \theta - gl \sin \theta) = F l \cos \theta
\]

\[
ml^2 \dot{\theta} + ml x \cos \theta + ml x \sin \theta = F l \cos \theta
\]