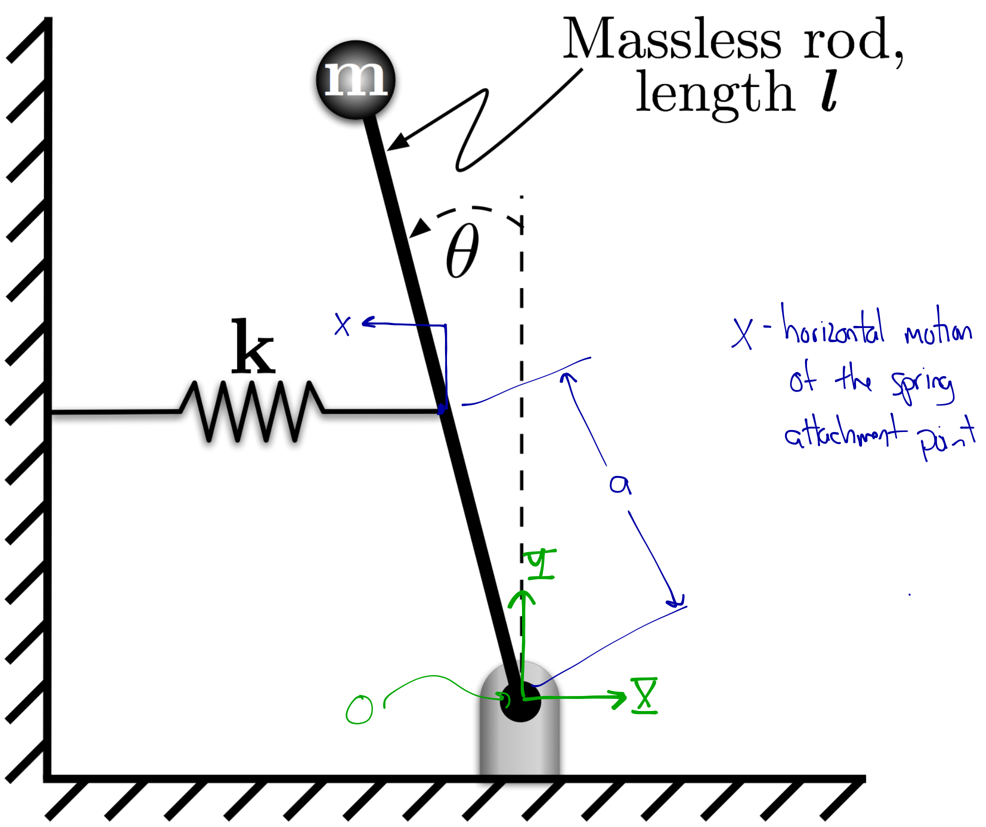
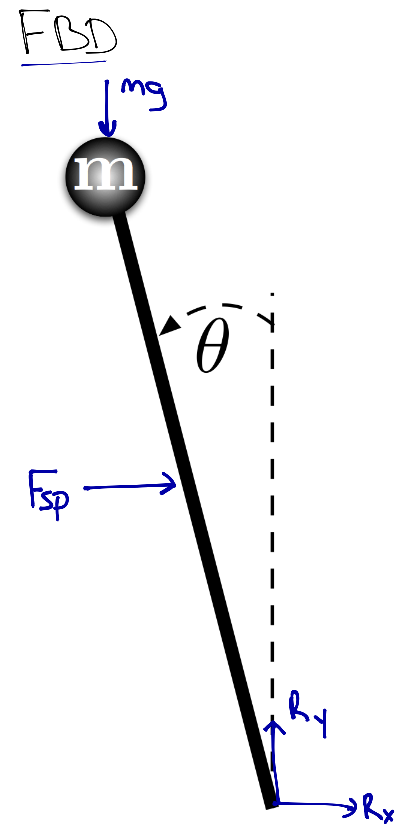
A Less-Simple Example (Newton/Euler)





Q: What is F_{sp}?

Look at the horizontal displacement (assuming small angles) of the connection point

$x = a\sin\theta$ } If $\theta = 0$ is the equil., then this is the δ of the spring

$F_{sp} = k\delta = ka\sin\theta$

Q: Where should we sum moments about?

point O — pure rotation about O ($\bar{a}_O = 0$) and $\bar{r}_{R/O} = 0$ so reactions don't contribute

$$I_O\bar{\alpha} = \sum \bar{M}_O = \left(\bar{r}_{sp/O} \times \bar{F}_{sp}\right) + \left(\bar{r}_{m/O} \times \bar{F}_{gr}\right)$$

$$= \left[\left(-a\sin\theta\,\bar{I} + a\cos\theta\,\bar{J}\right) \times \left(ka\sin\theta\,\bar{I}\right)\right] + \left[\left(-l\sin\theta\,\bar{I} + l\cos\theta\,\bar{J}\right) \times \left(-mg\,\bar{J}\right)\right]$$

$$I_O\ddot{\theta}\,\bar{K} = \left[-ka^2\cos\theta\sin\theta\,\bar{K}\right] + \left[mgl\sin\theta\,\bar{K}\right]$$

$$\boxed{I_O\ddot{\theta} + ka^2\cos\theta\sin\theta - mgl\sin\theta = 0}$$ ← Nonlinear   Q: How can we linearize?

Assume small angles, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ ← Also assumes $\theta_{eq} = 0$

$$I_O\ddot{\theta} + \left(ka^2 - mgl\right)\theta = 0 \longrightarrow I_O = ml^2 \longrightarrow ml^2\ddot{\theta} + \left(ka^2 - mgl\right)\theta = 0 \longrightarrow \boxed{\ddot{\theta} + \underbrace{\left(\frac{ka^2 - mgl}{ml^2}\right)}_{\omega_n^2}\theta = 0}$$ ← Linearized Eq. of Motion

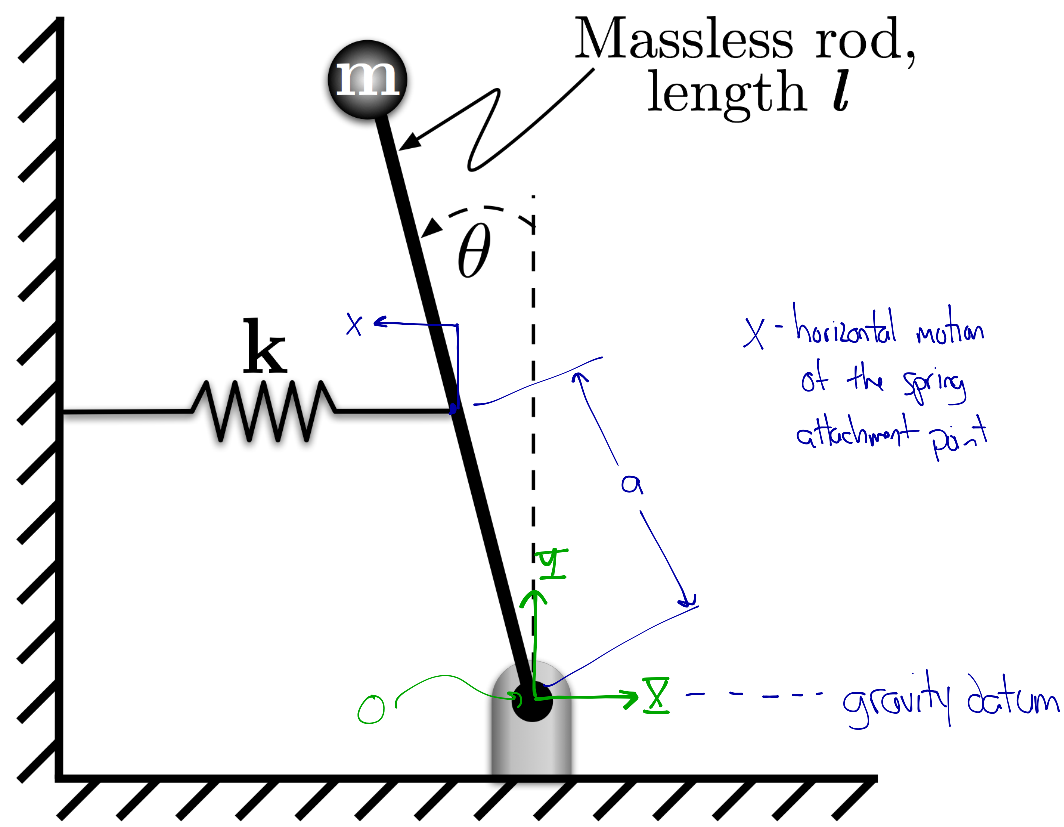
$$\omega_n^2 \longrightarrow \omega_n = \sqrt{\frac{ka^2 - mgl}{ml^2}}$$

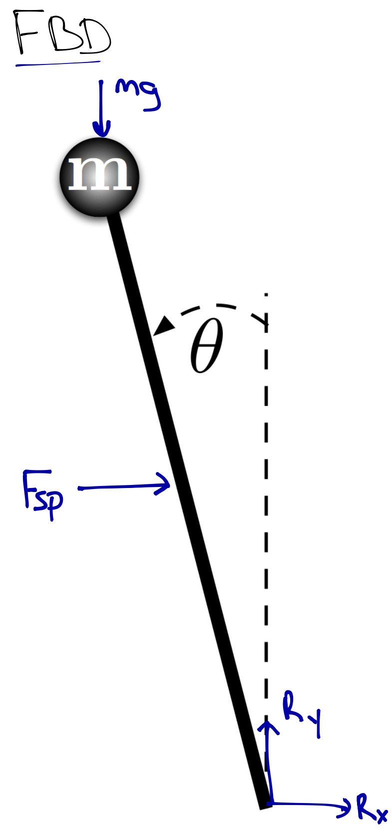
Q: What happens if the spring is too weak?

It falls over... the system is unstable — $\underbrace{\ddot{\theta} + \omega_n^2\theta = 0}_{\text{stable}}$ $\underbrace{\ddot{\theta} - \omega_n^2\theta = 0}_{\text{unstable}}$

Q: What is the critical value of k? — need $ka^2 > mgl \longrightarrow k > \frac{mgl}{a^2}$

A Less-Simple Example (Lagrange)





Q: What should we choose as the generalized coord? — θ

The system is in pure rotation about point O, so:

$$T = \frac{1}{2} I_O \dot{\theta}^2$$

The is potential energy from gravity and the spring

Set the gravity datum to the pin location $\longrightarrow$ $V_{gr} = mgl\cos\theta$

The spring potential is $\frac{1}{2}k\delta^2$. If we assume small angles $\delta \approx x$, so $V_{sp} = \frac{1}{2}kx^2$

But, we need to write all energies in terms of the generalized coords, so $x \approx a\sin\theta \longrightarrow V_{sp} = \frac{1}{2}k(a\sin\theta)^2$

$$V = \frac{1}{2}ka^2\sin^2\theta + mgl\cos\theta$$

$$L = T - V = \frac{1}{2}I_O\dot{\theta}^2 - \left(\frac{1}{2}ka^2\sin^2\theta + mgl\cos\theta\right)$$

Now, plug into Lagrange's Equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = I_O\dot{\theta} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I_O\ddot{\theta}$ $\qquad$ $\frac{\partial L}{\partial \theta} = -\left(ka^2\cos\theta\sin\theta - mgl\sin\theta\right)$

$$\boxed{I_O\ddot{\theta} + ka^2\cos\theta\sin\theta - mgl\sin\theta = 0}$$

$\longleftarrow$ Same as we got with Newton-Euler (It better be!)

From here, we could linearize as before.

Lagrange's Method with Viscous Damping

Remember that $F_D = c\dot{\delta}$ where $\dot{\delta}$ is the rate of change of the length of the damper

To match the form needed for Lagrange (remember, it's energy-based), integrate with respect to $\dot{\delta}$

$$\int F_D \, d\dot{\delta} = \int (c\dot{\delta}) \, d\dot{\delta} = \frac{1}{2}c\dot{\delta}^2$$ ← Looks similar to V_{sp}, right?

Rayleigh's Dissipation Function (RD) as:

$$RD \equiv \frac{1}{2}c\dot{\delta}^2$$ ← Note: Only valid for linear, viscous dampers

Use this to represent the damper in Lagrange's Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial RD}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad \text{for } i = 1, \ldots, n$$ ← Note: Still no external forces.

Lagrange's Method with External Forces

To understand external forces, we need to introduce...

Virtual Displacements

* Infinitesimally small changes in generalized coordinates
* Occur in zero time (no time elapses during the move)
* Do not violate system constraints

Consider $f(q_1, q_2, \ldots, t) = c$ as a system constraint. If we allow virtual displacements $\delta q_1, \delta q_2, \ldots$ then

$$f(q_1 + \delta q_1, q_2 + \delta q_2, \ldots, t) = c$$

constraint still holds

No time has passed

Why does this help? → It gives us a way to determine virtual work

Why does that help? → It tells us how to determine the contribution of each external force in the direction of each generalized coord.

Principle of Virtual Work

$$\delta W = \sum_j \bar{F}_j \cdot \delta \bar{r}_j + \sum_n \bar{M}_n \cdot \delta \bar{\theta}_n$$

Match terms between these two

$\delta \bar{r}_j$ = virtual displacement of point of application of $\bar{F}_j$

$\delta \bar{\theta}_n$ = virtual rotation of body on which $\bar{M}_n$ acts

And

$$\delta W_i = Q_i \delta q_i$$

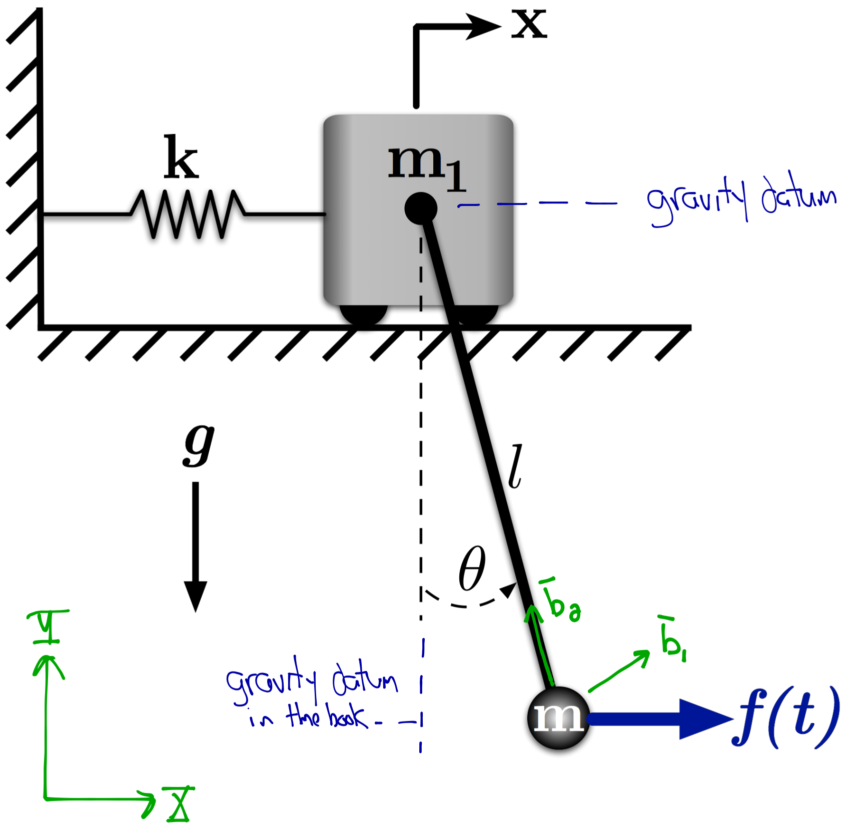
Q_i = component of the external force acting in direction δq_i ← component of force that causes motion in q_i

This goes into Lagrange's Eq

Now, we can write the "total" form of Lagrange's Equation:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial RD}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \quad \text{for } i = 1, 2, \ldots, n}$$

An Even-Less-Simple Example (Ex. 1.14-1.15)



2 bodies, 2 DOF → 2 generalized coords and 2 Eq of Motion

Use generalized coords (x, θ)

The velocity of m_1 is $\dot{x}$.

Q: What is the velocity of m?

It depends on both x and θ

$$\bar{V}_2 = \dot{x}\bar{I} + l\dot{\theta}\bar{b}_2 \quad \leftarrow \quad \bar{b}_2 = \cos\theta\bar{I} + \sin\theta\bar{J}$$

$$= (\dot{x} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}$$

Q: What should we choose as the gravity datum?

Let's choose the attachment point

(The book chooses the "lowest" pendulum point)

Now, form the energies, then the Lagrangian.

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m\bar{V}_2^T\bar{V}_2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[(\dot{x}+l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right] \cdot \left[(\dot{x}+l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right]$$

$\left(\text{or } \frac{1}{2}m\bar{V}_2 \cdot \bar{V}_2\right)$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m\left[(\dot{x}+l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right] \quad \leftarrow \text{Total kinetic energy}$$

$$V = V_{gr} + V_{sp} \qquad V_{gr} = mgh = -mgl\cos\theta \qquad \text{and} \qquad V_{sp} = \frac{1}{2}kx^2$$

$$V = -mgl\cos\theta + \frac{1}{2}kx^2 \quad \leftarrow \text{Total potential energy}$$

$$L = T - V \quad \leftarrow \text{Lagrangian}$$

Q: What about the external force, f(t)? ← We have to find the Q_i terms for Lagrange's Eq.

To do so, we'll use virtual displacements and virtual work

We know that $\bar{V}_2 = (\dot{x} + l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}$ ← This is the point the force is applied. We'll use this velocity to figure out the virtual displacements.

$$\frac{d\bar{r}_2}{dt} = \left(\frac{dx}{dt} + l\frac{d\theta}{dt}\cos\theta\right)\bar{I} + \left(l\frac{d\theta}{dt}\sin\theta\right)\bar{J} \quad \leftarrow \text{multiply by } dt \text{ and replace } dq_i \text{ with } \delta q_i$$

$$\delta\bar{r}_2 = (\delta x + l\delta\theta\cos\theta)\bar{I} + (l\delta\theta\sin\theta)\bar{J} \quad \leftarrow \text{virtual displacement of m}$$

An Even-Less-Simple Example (cont.)

To find Q_i, we need to find the virtual work done by the force $\left(= \text{Force} \cdot \text{virtual displacement}\right)$

$$\delta W = f\bar{I} \cdot \delta \bar{r}_2 = f\bar{I} \cdot \left[\left(\delta x + l\delta\theta\cos\theta\right)\bar{I} + \left(l\delta\theta \sin\theta\right)\bar{J} \right] = F\left(\delta x + l\delta\theta\cos\theta\right)$$

$$\delta W = \underbrace{F}_{Q_1}\underbrace{\delta x}_{\delta q_1} + \underbrace{\left(F l\cos\theta\right)}_{Q_2}\underbrace{\delta\theta}_{\delta q_2}$$

Now, apply Lagrange's Eq for each q_i - $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$

$q_1 = x$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_1$$

$$\frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + \frac{1}{2}m\left[2\left(\dot{x} + l\dot{\theta}\cos\theta\right)\right] \qquad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) \qquad Q_1 = f$$

$$\boxed{\left(m_1+m\right)\ddot{x} + ml^2\ddot{\theta}\cos\theta - ml^2\dot{\theta}^2\sin\theta + kx = F}$$

$q_2 = \theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_2$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left[2\left(\dot{x} + l\dot{\theta}\cos\theta\right)\left(l\cos\theta\right) + 2\left(l\dot{\theta}\sin\theta\right)\left(l\sin\theta\right)\right]$$

$$= m\left[l\dot{x}\cos\theta + l^2\dot{\theta}\cos^2\theta + l^2\dot{\theta}\sin^2\theta\right] = m\left(l\dot{x}\cos\theta + l^2\dot{\theta}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left(l\ddot{x}\cos\theta - l\dot{x}\dot{\theta}\sin\theta + l^2\ddot{\theta}\right)$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}m\left[2\left(\dot{x} + l\dot{\theta}\cos\theta\right)\left(-l\dot{\theta}\sin\theta\right) + 2\left(l\dot{\theta}\sin\theta\right)\left(l\dot{\theta}\cos\theta\right)\right] - mgl\sin\theta$$

$$= m\left[-l\dot{x}\dot{\theta}\sin\theta - gl\sin\theta\right]$$

$$Q_2 = fl\cos\theta$$

$$m\left(l\ddot{x}\cos\theta - \cancel{l\dot{x}\dot{\theta}\sin\theta} + l^2\ddot{\theta}\right) - m\left(\cancel{-l\dot{x}\dot{\theta}\sin\theta} - gl\sin\theta\right) = fl\cos\theta$$

$$\boxed{ml^2\ddot{\theta} + ml\ddot{x}\cos\theta + mgl\sin\theta = fl\cos\theta}$$