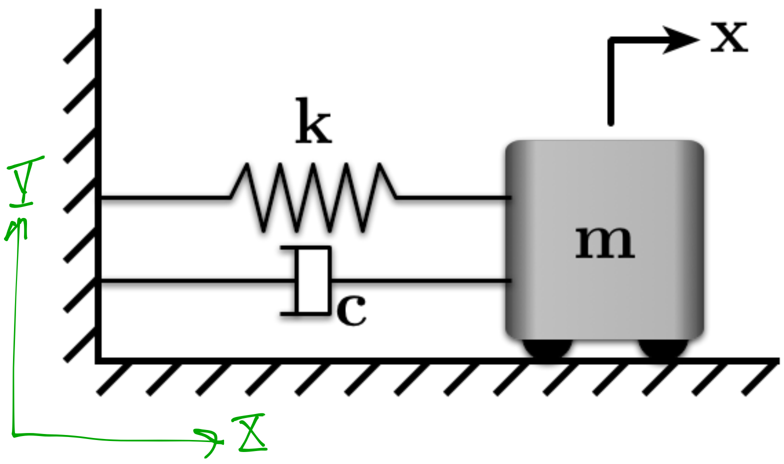
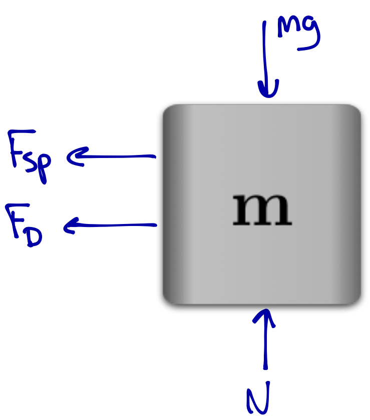
Viscous Damping (Sec. 1.4)



We generally assume viscous damping

$F_{damper} = c\dot{\delta}$ ← force is proportional to the rate of change of the length

FBD



$m\ddot{x} = -F_{sp} - F_D$

$m\ddot{x} = -kx - c\dot{x}$

$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

To find $x(t)$, use the same solution procedure as the undamped case.

Assume $\underbrace{x(t) = ae^{\lambda t} \quad \left(\text{so } \dot{x}(t) = a\lambda e^{\lambda t} \text{ and } \ddot{x}(t) = a\lambda^2 e^{\lambda t}\right)}_{\text{plug these into the eq. of motion}}$

$m\left(a\lambda^2 e^{\lambda t}\right) + c\left(a\lambda e^{\lambda t}\right) + k\left(ae^{\lambda t}\right) = 0$

$\underbrace{\left(m\lambda^2 + c\lambda + k\right)}_{=0 \text{ to have a nontrivial solution}} ae^{\lambda t} = 0$

$m\lambda^2 + c\lambda + k = 0$

$\lambda_{1,2} = \frac{1}{2m}\left(-c \pm \sqrt{c^2 - 4mk}\right)$ ← Not a very "friendly" form

Viscous Damping (cont.)

Look back at the equation of motion — $m\ddot{x}+c\dot{x}+kx=0$ ← divide entire equation by m

$$\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=0$$

Let $\frac{c}{m}=2\zeta\omega_n$ $\frac{k}{m}=\omega_n^2$

$\zeta \equiv$ damping ratio

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=0$$

Now, the solution depends on ζ:

If $\zeta>1$: overdamped

- no oscillation
- 2 negative real roots

$$\lambda_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

If $\zeta<1$ - underdamped

- oscillates
- complex conjugate roots

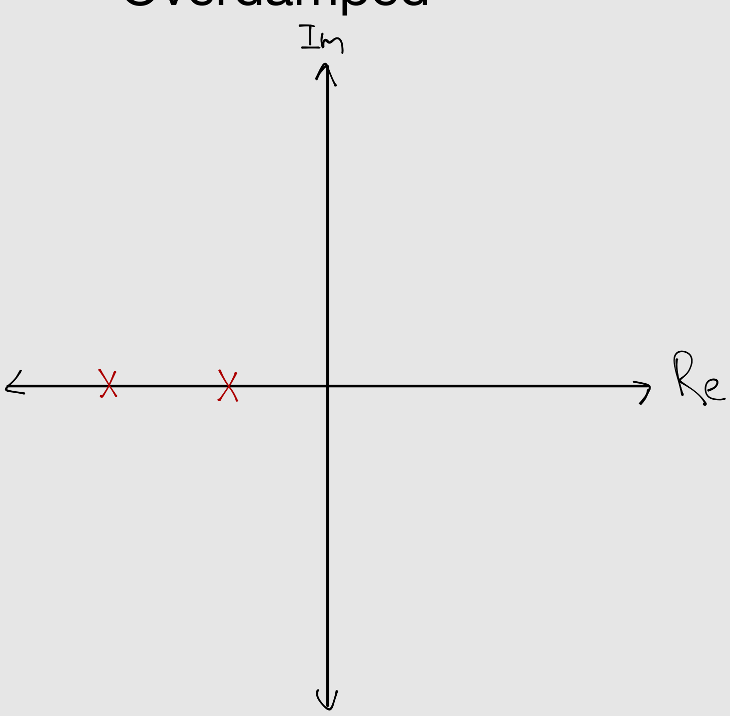
$$\lambda_{1,2}=-\zeta\omega_n \pm i\underbrace{\omega_n\sqrt{1-\zeta^2}} = -\zeta\omega_n \pm i\omega_d$$

damped natural freq. - $\omega_d=\omega_n\sqrt{1-\zeta^2}$

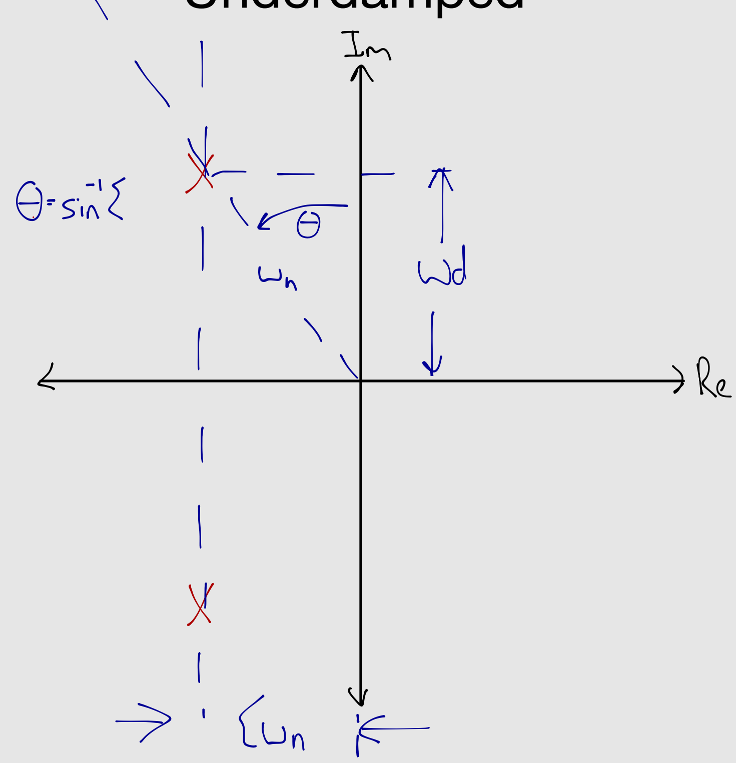
Controls Preview/Review

Plotting the system poles on the real/imaginary axis gives us information about the system.

Overdamped

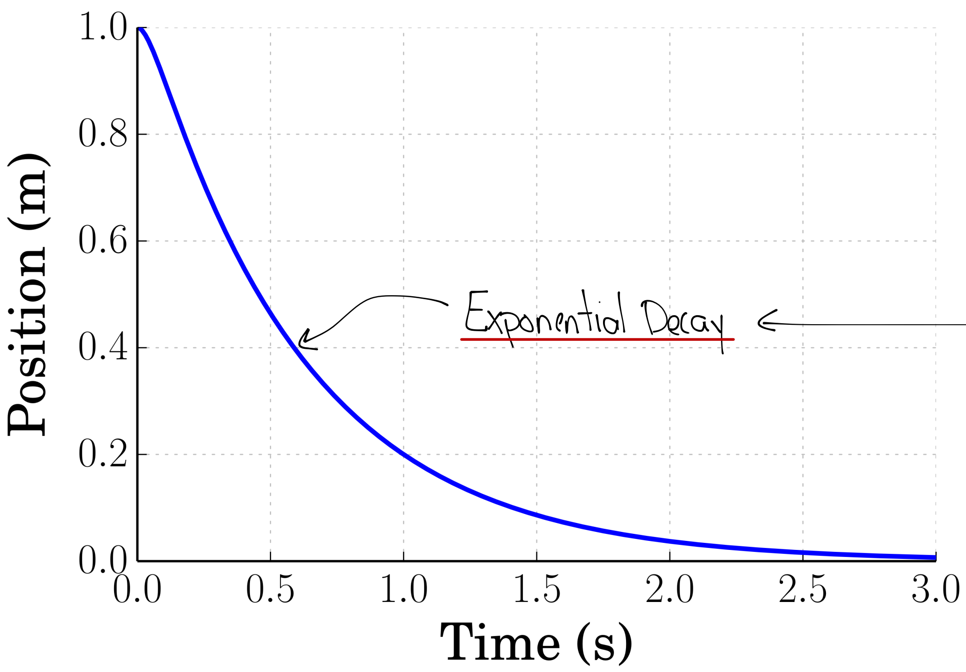


Underdamped



Overdamped Example Response

given $x(0) = 1$ and $\dot{x}(0) = 0$



goes to 0 as $t \to \infty$

Q: When might an overdamped response be desirable? Not desirable?

Desirable for no vibration or overshoot is desired

* automatic doors
* chemical processes

Undesirable:

* Car suspensions (uncomfortable)
* Controlled mechanical systems (too slow)

Underdamped Example Response

Start by looking at the roots

$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \longrightarrow x(t) = ae^{\lambda t}$, so:

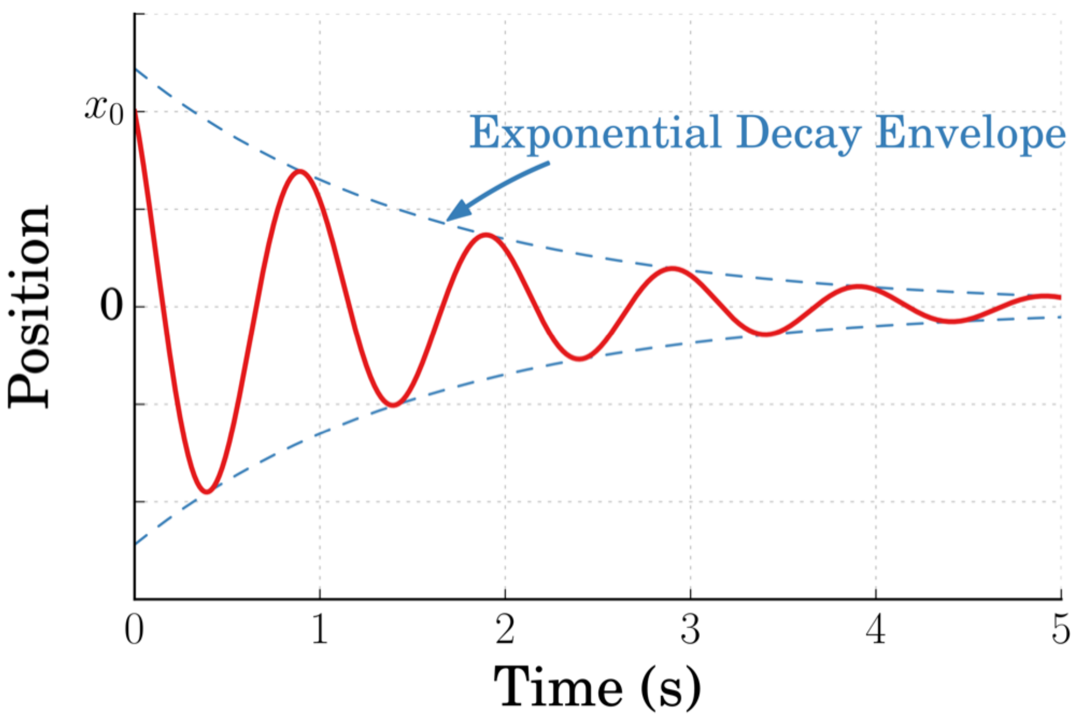
$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} = a_1 \exp\left[\left(-\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2}\right)t\right] + a_2 \exp\left[\left(-\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2}\right)t\right]$$

$$= a_1 e^{-\zeta\omega_n t} e^{i\omega_d t} + a_2 e^{-\zeta\omega_n t} e^{-i\omega_d t} \longleftarrow \text{Remember } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Collect the terms:

$$x(t) = \underbrace{e^{-\zeta\omega_n t}}_{\text{decay envelope}} \underbrace{\left(a_1 e^{i\omega_d t} + a_2 e^{-i\omega_d t}\right)}_{\text{oscillatory terms}} \longleftarrow \text{We can also write this in terms of sines and cosines}$$

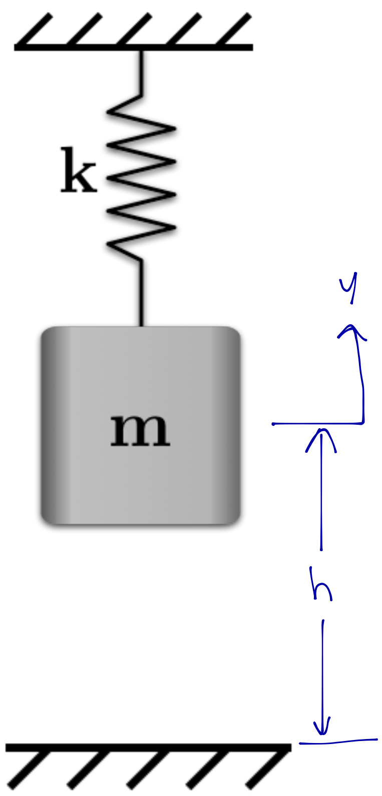
$$x(t) = \underbrace{e^{-\zeta\omega_n t}}_{\text{decay envelope}} \underbrace{\left[b_1\cos(\omega_d t) + b_2 \sin(\omega_d t)\right]}_{\text{oscillatory terms}}$$



Energy

Kinetic Energy - energy of motion

Potential Energy - stored energy



Linear Kinetic

$T = \frac{1}{2} m \bar{v}^T \bar{v} = \frac{1}{2} m \bar{v} \cdot \bar{v}$ ← vector form

Remember that $\bar{v}$ is the total linear velocity of m

more often see $T = \frac{1}{2} m v^2$

Gravitational Potential

$V_{gr} = mgh$

$h \equiv$ height above some datum (that we choose)

at that datum $V_{gr} = 0$ ← so "ground" or the lowest point is often a good choice

Q: What about the spring potential?

Spring Potential

$V_{gr} = \frac{1}{2} k \delta^2$ δ = spring deflection from equil.

Note: If a mass is in pure rotation, we can use a simplified version of the kinetic energy

$T = \frac{1}{2} \bar{\omega} \cdot \bar{H}_o \longrightarrow \frac{1}{2} I_o \omega^2$ } Remember that this is a special case. Be careful when using

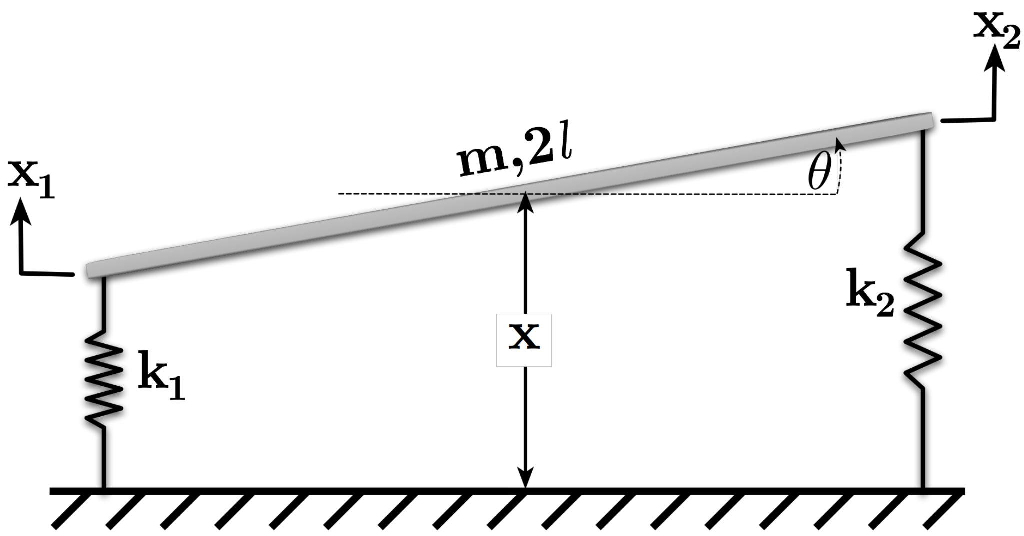
Lagrange's Equations/Method (Sec. 1.5)

* Energy-based method

* Allows us to ignore internal/interaction forces (if we want to)

* Usually based around:
  - generalized coordinates ← minimal set of independent coords needed to describe the system
  - virtual displacements ← Used in derivation and for external forces

Generalized Coordinates



We can choose:

(x, θ) or (x_1, x_2)

Generalized coords are often written as:

$\bar{q} = q_1, q_2, \ldots, q_n$

Here: $\bar{q} = (x, \theta)$ or $\bar{q} = (x_1, x_2)$

Lagrange's Equation (with no external forces or damping)

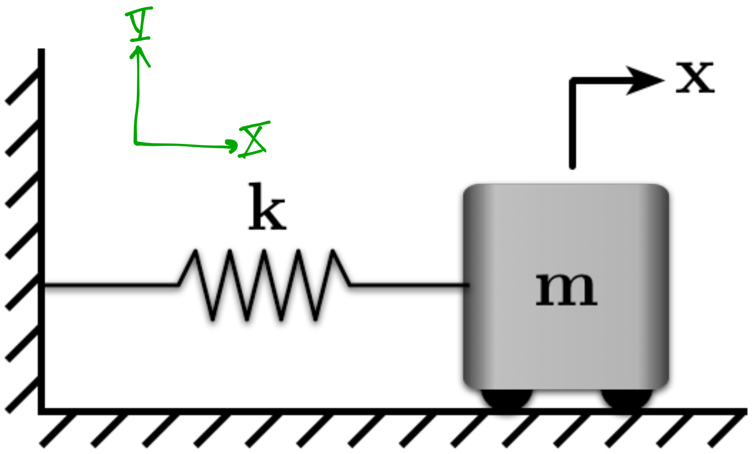
Define $L \equiv T - V$ = Kinetic Energy − Potential Energy ← The Lagrangian

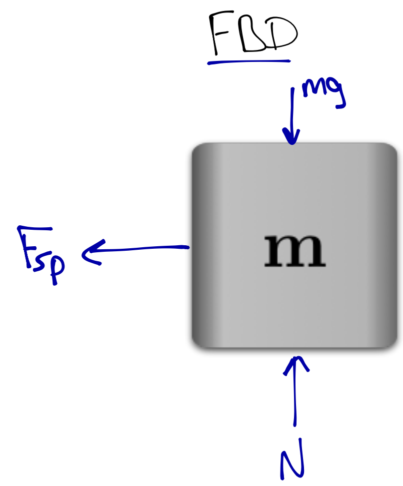
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad , \quad i = 1, 2, \ldots, n$$

where n is the number of generalized coords (= # of DOF)

Note: Be sure to include all system energies and define them consistently

Simple Linear Example of Lagrange's Method





Q: What should our generalized coordinate be?

$q_1 = x$

Now, form the kinetic and potential energies in terms of the generalized coordinates.

$T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$ so $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

Now, plug into Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

First term

$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$

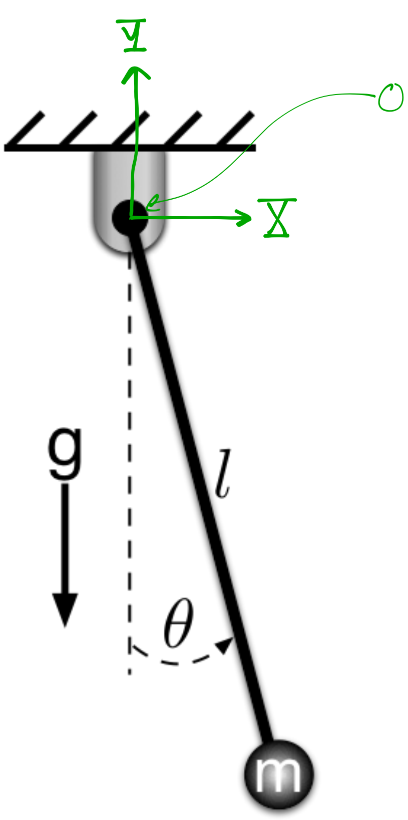
Second Term

$\frac{\partial L}{\partial x} = -kx$

$m\ddot{x} - (-kx) = 0$

$\boxed{m\ddot{x} + kx = 0}$ $\longleftarrow$ This matches what we got with Newton/Euler. It better!!!

Simple Rotational Example of Lagrange's Method



Q: What should we choose as a generalized coord? — θ

Q: Why not (x, y) of m? — They are not independent

System is in pure rotation about O, so we can use the "special" form of the rotational energy

$T = \frac{1}{2}I_o\dot{\theta}$

Define the gravity datum as point O, so $V_{gr} = mg(-l\cos\theta)$

$L = T - V = \frac{1}{2}I_o\dot{\theta}^2 + mgl\cos\theta \longleftarrow$ plug into $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = I_o\dot{\theta}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I_o\ddot{\theta}$, $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$

$I_o\ddot{\theta} + mgl\sin\theta = 0 \longrightarrow \boxed{ml^2\ddot{\theta} + mgl\sin\theta = 0}$