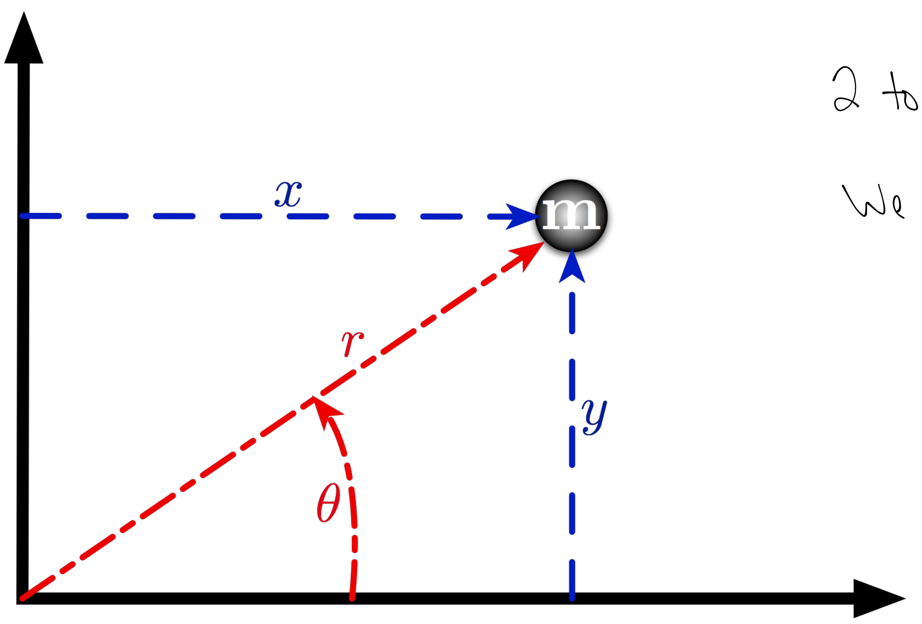
Degrees of Freedom (DOF)

Q: What is a degree-of-freedom?
- informally, the ways in which the system can move
- more formally, the minimum number of independent variables to fully describe the system configuration

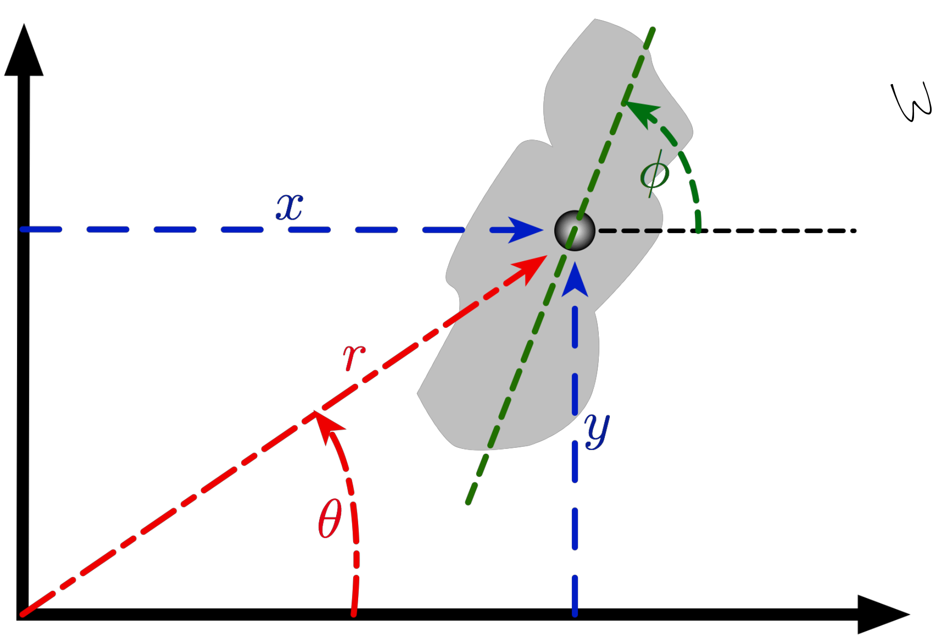
Q: How many DOF does a point mass in a plane have?



2 to fully describe it

We can choose (x,y) or (r,θ) or ...

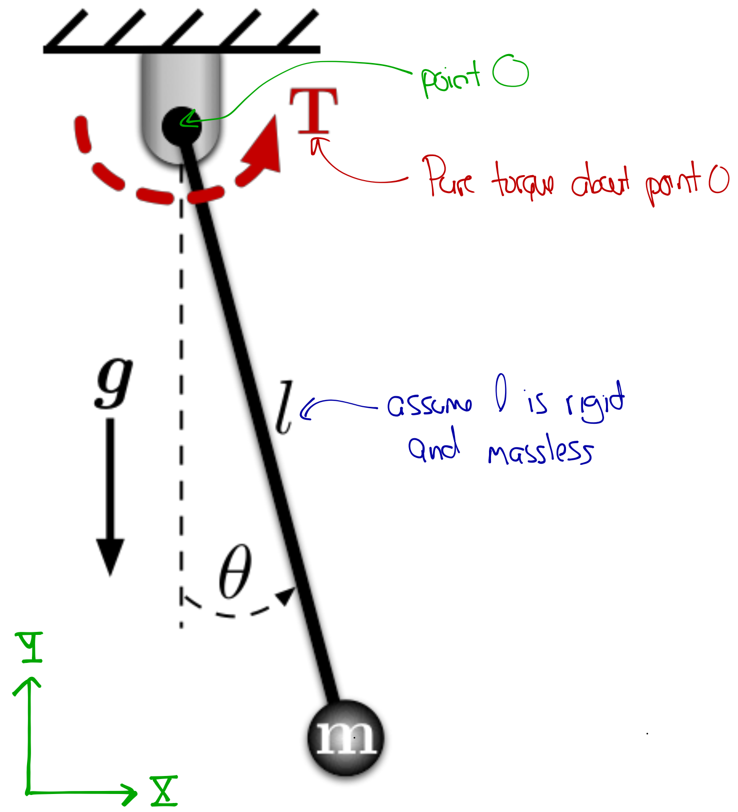
Q: What about a rigid-body in a plane?



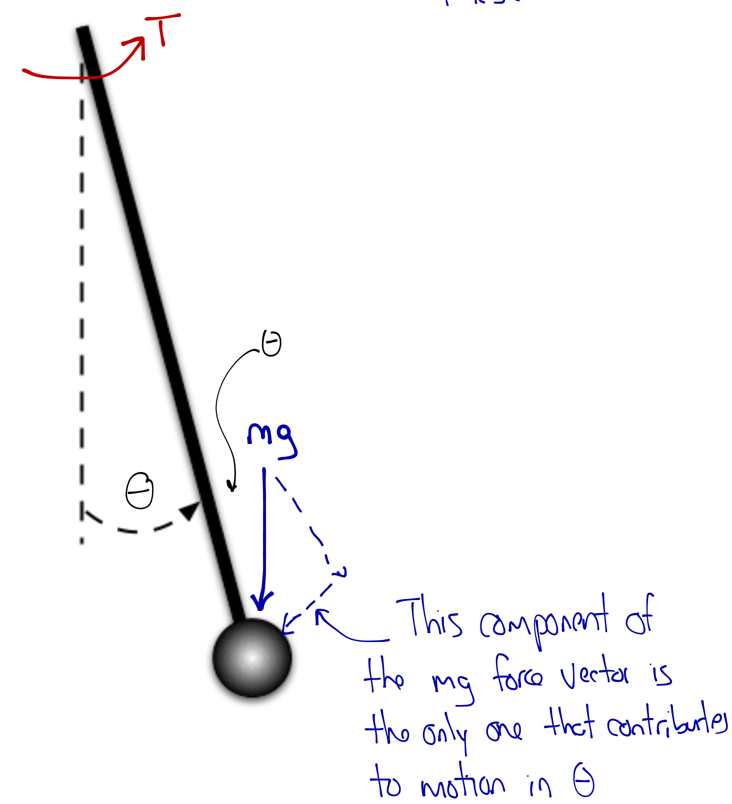
We need 3:

(x,y,ϕ) or (r,θ,ϕ) are possible choices

Another Rotational Example



Free Body Diagram ← ALWAYS draw these



Q: Where should we sum moments about?

point O - $\bar{a}_o = 0$ (system is in pure rotation about O)

$$I_o \bar{\alpha} = \Sigma \bar{M}_o$$ ← I_o is the moment of inertia about O. This is the rotational equiv. of mass

$$= \bar{T} + \left(\bar{r}_{m/o} \times -mg\bar{J}\right)$$

$$= T\bar{K} + \left[\left(l\sin\theta\bar{I} - l\cos\theta\bar{J}\right) \times -mg\bar{J}\right]$$

$$I_o\bar{\alpha} = T\bar{K} + \left(-mgl\sin\theta\right)\bar{K}$$

Q: What is I_o for this system?

m is a point mass at distance l from O ⟶ $I_o = ml^2$

Q: What is $\bar{\alpha}$ in this case? ← $\bar{\alpha} = \ddot{\theta}\bar{K}$ (pure rotation using right-hand rule)

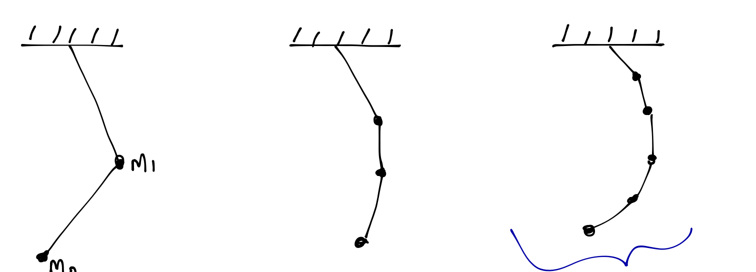
So,

$$ml^2\ddot{\theta} = T - mgl\sin\theta$$ ← all in $\bar{K}$ direction

$$ml^2\ddot{\theta} + mgl\sin\theta = T$$

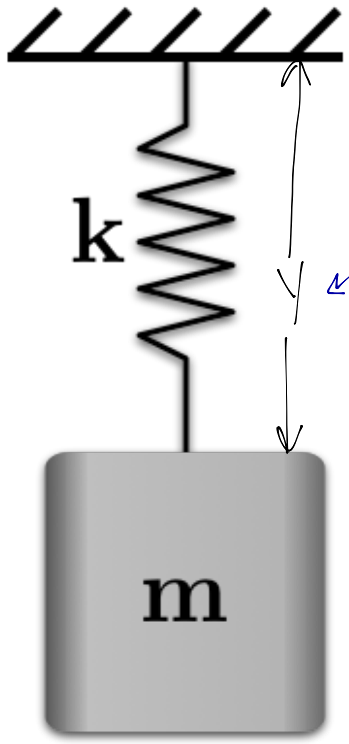
$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T}{ml^2}$$

Q: We assumed l was rigid and massless. What if that's not true?



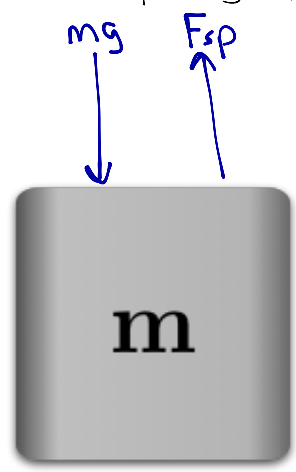
Basic idea behind FEA

Chapter 1 – Free Vibration of 1DOF Systems



Notice that this y is the absolute measure of distance from the hanger/ground

Free Body Diagram



Now, just apply $\Sigma \bar{F} = m\bar{a}$

Remember: positive y is downward here, based on how it's defined

$\Sigma \bar{F} = -F_{sp} + mg \qquad \bar{a} = \ddot{y}$

$m\ddot{y} = -F_{sp} + mg$

Q: What is F_{sp}?

$F_{sp} = k\delta$ — $\delta \equiv$ deflection from equilibrium

Because of how we defined y in this problem (as an absolute measure), we need to define $\delta = (y - \ell)$ where ℓ is the spring's equilibrium length

$F_{sp} = k(y - \ell)$

$$\boxed{m\ddot{y} = -k(y - \ell) + mg}$$

This is a valid equation of motion. However, it's not the "friendliest" ODE to solve.

We can do better by realizing that free vibration will occur around the equil. point.

Q: What is the equilibrium position of this system?

Occurs where the spring and gravity forces balance

$$\cancel{m\ddot{y}} = -k(y-\ell) + mg \longrightarrow mg = k(y_0-\ell) \text{ defines the equil. position } y_0$$

$\ddot{y}=0$ at equil.

Now, let $y(t) = y_0 + x(t) \leftarrow x(t)$ is motion around the equilibrium position

$(\ddot{y}(t) = \ddot{x}(t)$ because $y_0 =$ constant$)$

Substitute this definition into the equation of motion.

$$m\ddot{x} = -k\left[(y_0+x)-\ell\right] + mg = -k\left[(y_0-\ell)+x\right] + \underbrace{mg}$$

We know that $mg = k(y_0-\ell)$

$$m\ddot{x} = \cancel{-k(y_0-\ell)} - kx + \cancel{k(y_0-\ell)}$$

$$m\ddot{x} = -kx$$

$$\boxed{m\ddot{x} + kx = 0}$$

Q: How do we find the response, $x(t)$?

It is a $\underbrace{\text{linear}}_{\text{only } x \text{ and its deriv.}}$, $\underbrace{\text{constant coefficient}}_{m \text{ and } k \text{ are constant}}$, $\underbrace{\text{autonomous}}_{\text{No forcing function}}$ ODE

So, we know the solution has the form:

$x(t) = ae^{\lambda t} \leftarrow$ Plug this "assumed" solution into the ODE

$$\dot{x} = a\lambda e^{\lambda t} \qquad \ddot{x} = a\lambda^2 e^{\lambda t}$$

$$m(a\lambda^2 e^{\lambda t}) + k(ae^{\lambda t}) = 0$$

$(m\lambda^2 + k)ae^{\lambda t} = 0 \leftarrow$ for this equation to be true $m\lambda^2 + k = 0$ $(a \neq 0$ and $e^{\lambda t} \neq 0$ for finite $t)$

Solve for λ: $m\lambda^2 + k = 0 \rightarrow \lambda^2 = \frac{-k}{m} \longrightarrow \lambda = \pm i\sqrt{\frac{k}{m}}$, so the solution is

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

$$\boxed{x(t) = a_1 e^{-i\sqrt{\frac{k}{m}}t} + a_2 e^{i\sqrt{\frac{k}{m}}t}} \longleftarrow \text{Need to solve for } a_1 \text{ and } a_2 \text{ based on initial conditions}$$

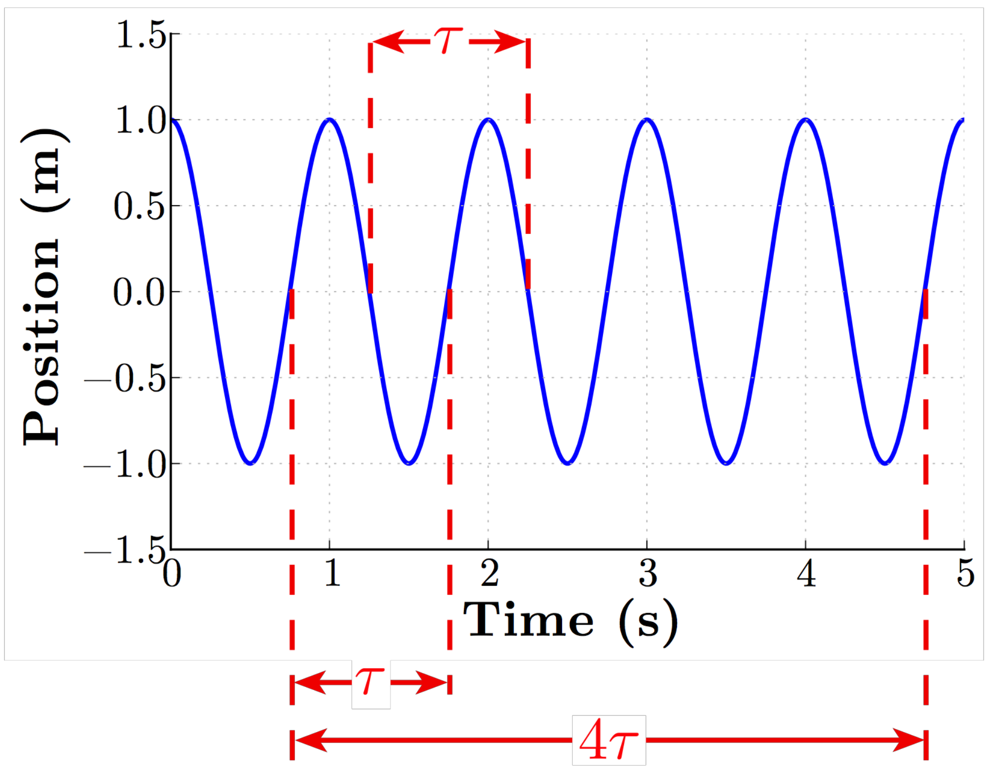
Q: How else could we write this equation?

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t) \quad \text{so}$$

$$x(t) = a_1 e^{-i\sqrt{\frac{k}{m}}t} + a_2 e^{i\sqrt{\frac{k}{m}}t} = b_1\cos\left(\sqrt{\tfrac{k}{m}}t\right) + b_2\sin\left(\sqrt{\tfrac{k}{m}}t\right)$$

Q: What's special about $\sqrt{\frac{k}{m}}$ for this system/solution?

It's the natural frequency of this system. $\omega_n = \sqrt{\frac{k}{m}}$ ← system vibrates at this frequency



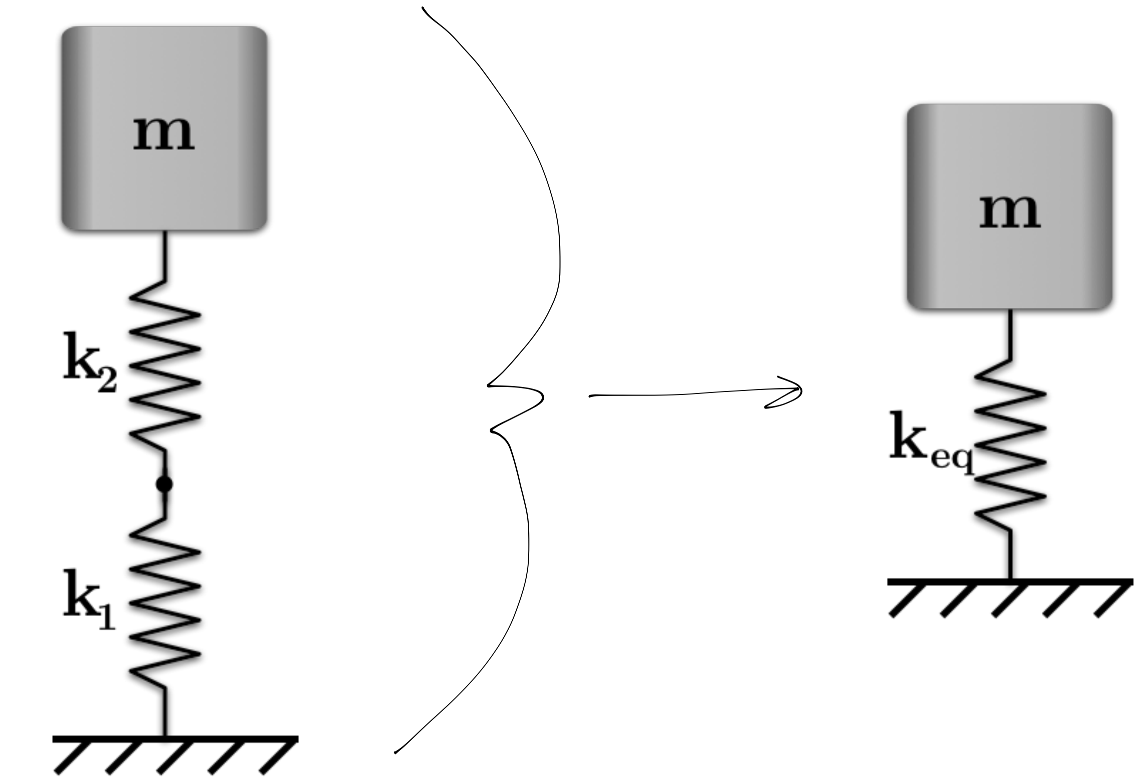
τ = period of vibration (s)

$$\omega_n = \frac{2\pi}{\tau} \qquad \tau = \frac{2\pi}{\omega_n}$$

Note: ω_n units are rad/s
(to get Hz, divide by 2π)

In some fields/places, it is convention to use ω_n for natural freq. in rad/s and f to represent natural freq. in Hz.

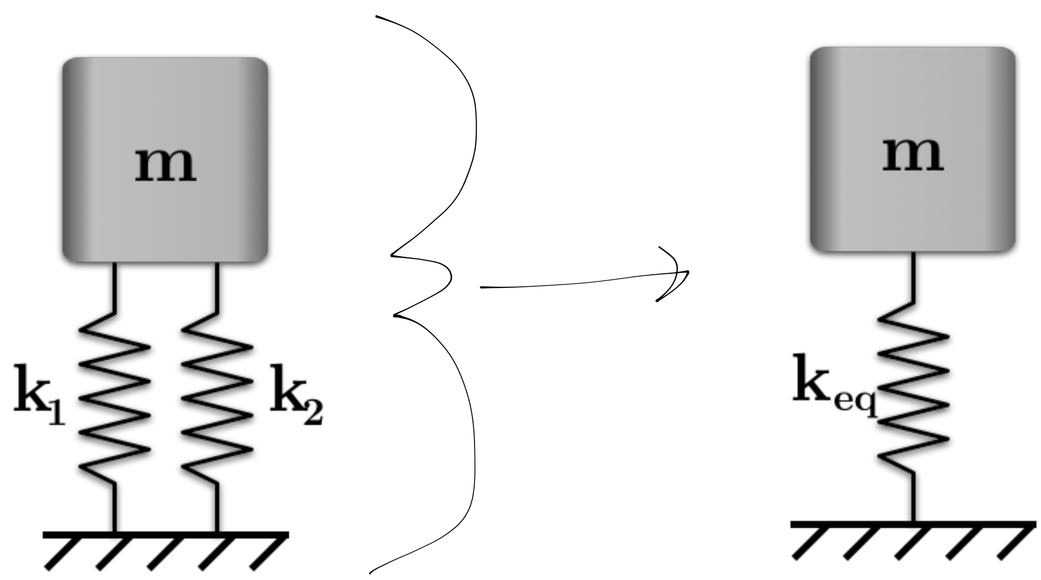
Equivalent Springs



Springs in Series

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \longrightarrow k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

Notice that springs in series are like resistors in parallel



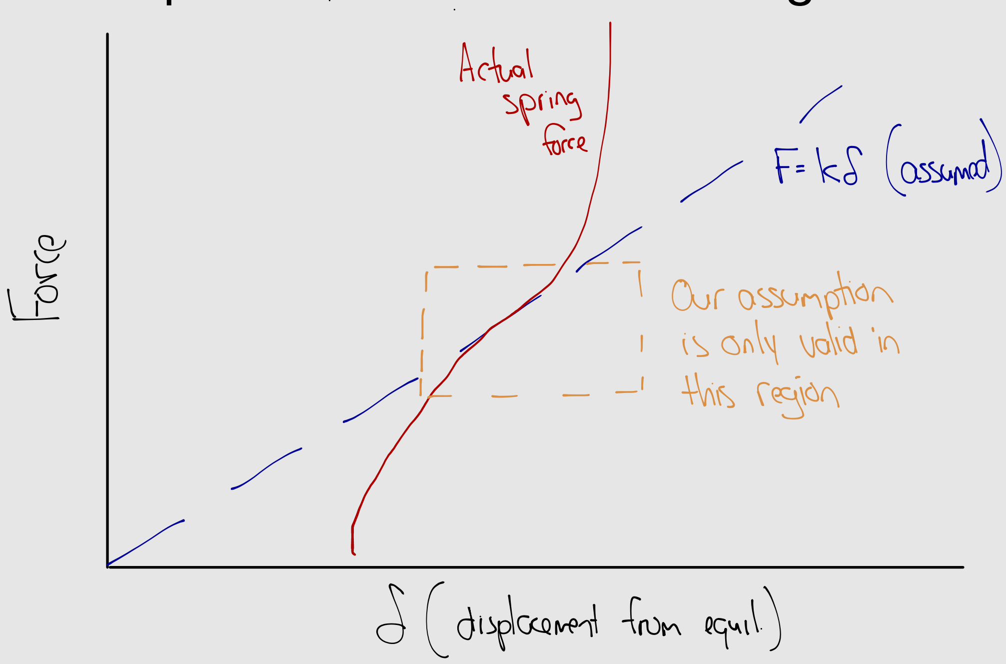
Springs in Parallel

$$k_{eq} = k_1 + k_2$$

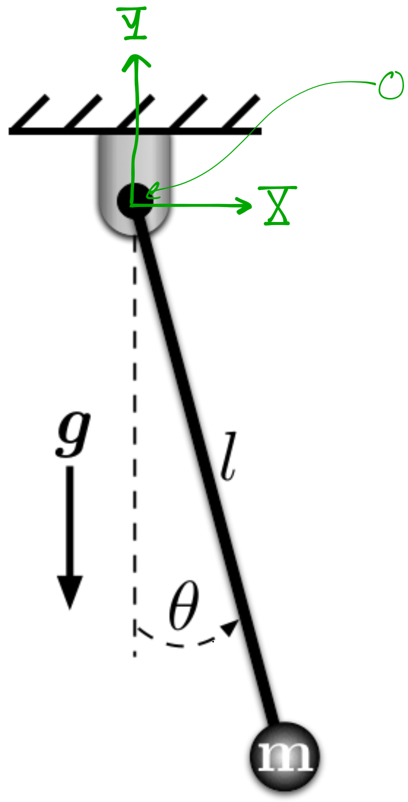
Like resistors in series.

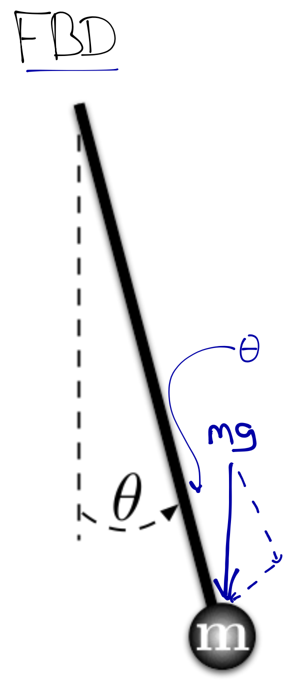
Aside:

$F_{sp} = k\delta$ assumes that the spring is linear. We can usually only make this assumption within some small region.



Rotational Vibration & Linearization (Sec. 1.3)





Sum moments about point O to find:

$$I_o\bar{\alpha} = \Sigma \bar{M}_o = \left(\bar{r}_{m/o} \times -mg\bar{J}\right) = \left(l\sin\theta\,\bar{I} - l\cos\theta\,\bar{J}\right) \times \left(-mg\bar{J}\right)$$

$$I_o\ddot{\theta}\bar{K} = -mgl\sin\theta\,\bar{K}$$

$I_o = ml^2$ (point mass m in pure rotation at a distance l)

$$ml^2\ddot{\theta} = -mgl\sin\theta \longrightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Q: How do we find $\theta(t)$?

This diff eq. is nonlinear (has $\sin\theta$ instead of θ), so:

1) complex diff. eq. solution procedure or

2) linearize

Linearization

Typically, we linearize about some operating point. In many cases, it makes sense to linearize about an equilibrium position.

Q: How do we know what the equilibrium positions are?

One "trick" is to eliminate the "motion" variables (velocity and higher order derivatives) from the equation of motion.

$$\underbrace{ml^2\ddot{\theta}}_{\text{set motion term}=0} + \underbrace{mgl}_{mgl\neq 0}\underbrace{\sin\theta}_{\text{so this term must equal 0 for this equation to hold}} = 0 \longrightarrow \sin\theta = 0 \text{ is the equil position}$$

$$\downarrow$$

$\theta = n\pi$, where n is an integer

$\theta = 0$ $\quad$ $\theta = \pi$ $\leftarrow$ 2 equilibrium positions

Q: Two equilibrium points... Which one should we choose?

It depends on the system and its operating conditions. In this case:

$\theta = 0 \longleftarrow$ cranes, pendulum-like systems

$\theta = \pi \longleftarrow$ rockets, segway, humans

Now, linearize about the chosen equil. There are many methods.

Taylor Series Expansion

$$\sin\theta \approx \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots$$

If we assume that θ is small, then:

$$\theta \gg -\frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots \longleftarrow \text{So, for small } \theta\text{, we can approx. } \sin\theta \approx \theta$$

Back to the pendulum problem (using this linearization):

$$ml^2\ddot{\theta} + mgl(\sin\theta) = 0 \longrightarrow ml^2\ddot{\theta} + mgl(\theta) = 0 \longrightarrow \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0} \longleftarrow \text{Linearized equation of motion}$$

Linearized using Taylor Series Expansion and small angle assumption

Q: What do you notice about this equation?

It has the same form as the mass-spring equation.

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \omega_n = \sqrt{\frac{g}{l}}$$

Key point: Many vibratory systems can be written in this form.