MCHE 485: Mechanical Vibrations
Spring 2019 – Homework 4

Assigned:	Thursday, March 28th
Due:	This assignment is not collected. Solutions will be posted on Monday, April 1st
Assignment:	From "Principles of Vibration" by Benson Tongue, write the equations of motion for the following problems as a system of first-order differential equations. Then, write each State-space form. If the equations are nonlinear, linearize them first: 2.39, 2.40, 2.57, 4.8 From "Principles of Vibration" by Benson Tongue, for the problems below, set up the problem up to the point of needing to actually solve the eigenvalue problem. 4.1, 4.2, 4.9, 4.12, 4.13
Submission:	No submission required. This assignment will not be collected.

Problem 2.39

2.39. Find the transfer function of support excitation y to response angle θ for the pendular system shown in Figure P2.38. Make sure to linearize your equations. The pendulum is of length l and the freely pivoted upper end of the pendulum is moved horizontally according to

$$y(t) = a\sin(\omega t)$$

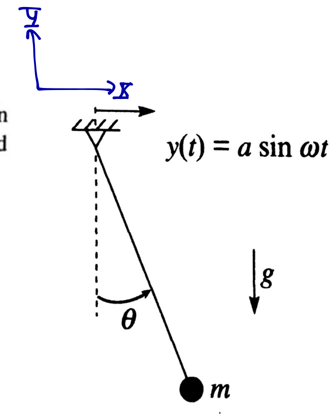


Figure P2.39

I will get the equations of motion using Lagrange's Method.

First, define the position of m

$$r_{m/o} = (y + l\sin\theta)\bar{I} - l\cos\theta\bar{J}$$

Using this write the velocity of m

$$\dot{r}_{m/o} = (\dot{y} + l\dot{\theta}\cos\theta)\bar{I} + l\dot{\theta}\sin\theta\bar{J} = \bar{v}_m$$

So,

$$T = \frac{1}{2}m v_m^T v_m = \frac{1}{2}m\left[(\dot{y}+l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right]\cdot\left[(\dot{y}+l\dot{\theta}\cos\theta)\bar{I} + (l\dot{\theta}\sin\theta)\bar{J}\right]$$

$$= \frac{1}{2}m\left[(\dot{y}+l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right] = \frac{1}{2}m\left[\dot{y}^2 + 2l\dot{y}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right]$$

$$= \frac{1}{2}m\left[\dot{y}^2 + 2l\dot{y}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right]$$

$$V = -mgl\cos\theta$$

$$L = \frac{1}{2}m\left[\dot{y}^2 + 2l\dot{y}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right] + mgl\cos\theta \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{2}m\left(2l\dot{y}\cos\theta + 2l^2\dot{\theta}\right) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml\ddot{y}\cos\theta - ml\dot{y}\dot{\theta}\sin\theta + ml^2\ddot{\theta}$$

$$= ml^2\ddot{\theta} - ml\dot{y}\dot{\theta}\sin\theta + ml\ddot{y}\cos\theta$$

$$\frac{\partial L}{\partial\theta} = -ml\dot{y}\dot{\theta}\sin\theta - mgl\sin\theta$$

$$ml^2\ddot{\theta} + ml\ddot{y}\cos\theta + mgl\sin\theta = 0 \longrightarrow$$ Assume small angles so $\sin\theta \approx \theta$ and $\cos\theta \approx 1$

$$ml^2\ddot{\theta} + ml\ddot{y} + mgl\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{-1}{l}\ddot{y} \longleftarrow$$ Linearized Equation of motion with $\omega_n = \sqrt{\frac{g}{l}}$

Now, assume $y(t) = a\sin\omega t$ so $\ddot{y}(t) = -\omega^2 a\sin\omega t$

assume $x(t) = \bar{x}\sin\omega t$ $\quad (\ddot{x} = -\omega^2\bar{x}\sin\omega t)$

Plug into the equation of motion $\rightarrow (-\omega^2 + \omega_n^2)\bar{x}\cancel{\sin\omega t} = \frac{\omega^2}{l}a\cancel{\sin\omega t} \rightarrow \boxed{\frac{\bar{x}}{a} = \frac{\omega^2}{l(\omega_n^2 - \omega^2)}}$

Problem 2.39 (cont.)

The equation of motion is:

$$ml^2\ddot{\theta} + mgl\sin\theta = -ml\ddot{y} \rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{1}{l}\ddot{y}$$

Define state-vector

$$\bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

Writing the equation in the form $\dot{\bar{w}} = f(\bar{w}, u, t)$:

$$\boxed{\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin\theta - \frac{1}{l}\ddot{y} \end{bmatrix} = \begin{bmatrix} w_2 \\ -\frac{g}{l}\sin w_1 - \frac{1}{l}\ddot{y} \end{bmatrix}}$$

We can write in state-space form by linearizing assuming small angles. The equation above becomes

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\theta - \frac{1}{l}\ddot{y} \end{bmatrix} = \begin{bmatrix} w_2 \\ -\frac{g}{l}w_1 - \frac{1}{l}\ddot{y} \end{bmatrix}$$

Now, write this in matrix form:

$$\boxed{\dot{\bar{w}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}\bar{w} + \begin{bmatrix} 0 \\ -\frac{1}{l} \end{bmatrix}\ddot{y}}$$

Problem 2.40

2.40. Find the transfer function between the displacement input y and the displacement output x for the system shown in Figure P2.40. $y = \bar{y}\sin(\omega t)$. The rigid bar pivots freely at O.

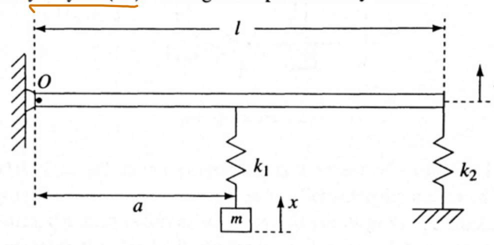
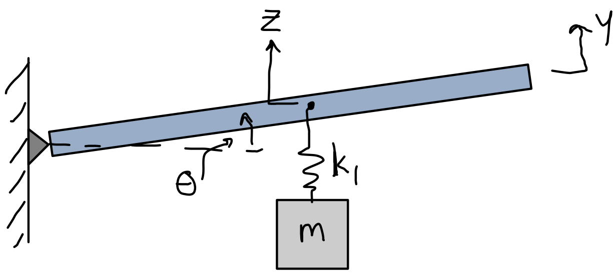


Figure P2.40

The problem specifies y(t), suggesting that we can control it exactly. This allows us to ignore this spring and any inertial effects of the bar. y(t) just becomes an input to the top of k1 via the bar.



Define z(t) as the vertical motion of the the top of k_1

y(t) is only defining what z(t) is

So, we need to relate y(t) to z(t). We can do this via their positions along the bar and θ.

We will assume that we are working about an equil. of $\theta = 0$

$y(t) \approx l\sin\theta \leftarrow$ at small angles $y(t) \approx l\theta$

so $\theta = \frac{y(t)}{l}$

Similarly,

$z(t) \approx a\sin\theta \leftarrow$ at small angles $z(t) \approx a\theta$

So, $z(t) = a\left(\frac{y(t)}{l}\right) = \frac{a}{l}y(t) = \frac{a}{l}\bar{y}\sin(\omega t)$ } This is a "seismic" input to the hanging mass-spring system

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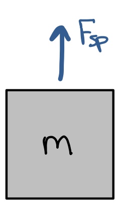

$F_{sp} = k(z-x)$ so

$m\ddot{x} = k(z-x)$

$m\ddot{x} + kx = kz \longrightarrow m\ddot{x} + kx = k\left(\frac{a}{l}\bar{y}\sin\omega t\right)$ or $\ddot{x} + \omega_n^2 = \omega_n^2\left(\frac{a}{l}\bar{y}\sin\omega t\right)$

Assume $x(t) = \bar{x}\sin\omega t$ (match the form of y(t))

$(-\omega^2 + \omega_n^2)\bar{x}\cancel{\sin\omega t} = \omega_n^2\left(\frac{a}{l}\bar{y}\cancel{\sin\omega t}\right)$

$$\boxed{\frac{\bar{x}}{\bar{y}} = \frac{\frac{a}{l}\omega_n^2}{\omega_n^2 - \omega^2}}$$

Problem 2.40 (cont.)

The equation of motion is:

$$m\ddot{x} + kx = \frac{ka}{l}Y \longrightarrow \ddot{x} = \frac{-k}{m}x + \frac{ka}{ml}Y$$

Define $\bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

$$\dot{\bar{w}} = \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{k}{m}x + \frac{ka}{ml}Y \end{bmatrix}$$

← This is a system of 1st order linear diff eq

$$\boxed{\dot{\bar{w}} = \begin{bmatrix} w_2 \\ -\frac{k}{m}w_1 + \frac{ka}{ml}Y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{ka}{ml} \end{bmatrix}Y}$$

← This is state-space form

Problem 2.57

2.57. Find the velocity $\dot{x}$ response for the system illustrated in Figure P2.57. $y = .01\sin(100t)$, $k = 8000$ N/m, $c_1 = 4$ N·s/m, $c_2 = 2$ N·s/m, $m = .25$ kg.

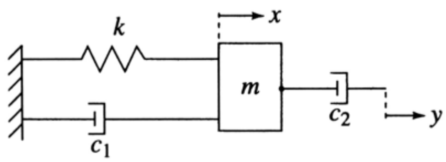

Figure P2.57

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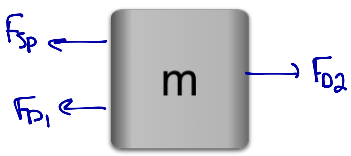

$F_{sp} = kx$ $\quad F_{D2} = c_2(\dot{y}-\dot{x})$

$F_{D1} = c\dot{x}$

So $m\ddot{x} = -kx - c_1\dot{x} + c_2(\dot{y}-\dot{x})$

$m\ddot{x} + (c_1+c_2)\dot{x} + kx = c_2\dot{y}$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{c_2}{m}\dot{y}$ where $2\zeta\omega_n = \frac{c_1+c_2}{m}$ and $\omega_n^2 = \frac{k}{m}$

write $y(t)$ as $\bar{y}e^{i\omega t}$ knowing that the "actual" input is the imaginary part of it $\Big\}$ $y(t) = \mathrm{Im}(0.01e^{i100t})$

Assume $x(t) = \bar{X}e^{i\omega t}$

$$(-\omega^2 + 2i\zeta\omega\omega_n + \omega_n^2)\bar{X}\cancel{e^{i\omega t}} = \frac{c_2}{m}(i\omega)\bar{Y}\cancel{e^{i\omega t}}$$

$$\bar{X} = \frac{(c_2/m)\,i\omega}{\omega_n^2-\omega^2+2i\zeta\omega\omega_n}\bar{Y}$$

So $$x(t) = \frac{(c_2/m)\,i\omega}{\omega_n^2-\omega^2+2i\zeta\omega\omega_n}\bar{Y}e^{i\omega t}$$

Look at this transfer function... mult numerator and denominator by $\omega_n^2-\omega^2-2i\zeta\omega\omega_n$ (the complex conjugate of the den.)

$$\bar{X} = \frac{\frac{c_2}{m}i\omega(\omega_n^2-\omega^2-2i\zeta\omega\omega_n)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2} \Longrightarrow \bar{X}_r = \frac{\frac{c_2}{m}2\zeta\omega^2\omega_n}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}$$

$$\bar{X}_c = \frac{\frac{c_2}{m}\omega(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)+(2\zeta\omega\omega_n)^2}$$

Problem 2.57 (cont.)

The equation of motion is

$$m\ddot{x} + (c_1+c_2)\dot{x} + kx = c_2\dot{y} \qquad \text{or} \qquad \ddot{x} = \frac{-k}{m}x - \frac{(c_1+c_2)}{m}\dot{x} + \frac{c_2}{m}\dot{y}$$

Define $\bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

So $$\dot{\bar{w}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{k}{m}x - \frac{(c_1+c_2)}{m}\dot{x} + \frac{c_2}{m}\dot{y} \end{bmatrix} = \begin{bmatrix} w_2 \\ -\frac{k}{m}w_1 - \frac{(c_1+c_2)}{m}w_2 + \frac{c_2}{m}\dot{y} \end{bmatrix}$$

$$\boxed{\dot{\bar{w}} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{(c_1+c_2)}{m} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{c_2}{m} \end{bmatrix} \dot{y}}$$

Problem 4.1

4.1. Find the two natural frequencies and their associated eigenvectors for the system illustrated in Figure P4.1. $m_1 = 1 \times 10^{-3}$ kg, $m_2 = 10 \times 10^{-3}$ kg, $k_1 = 3 \times 10^3$ N/m, $k_2 = 3 \times 10^3$ N/m.

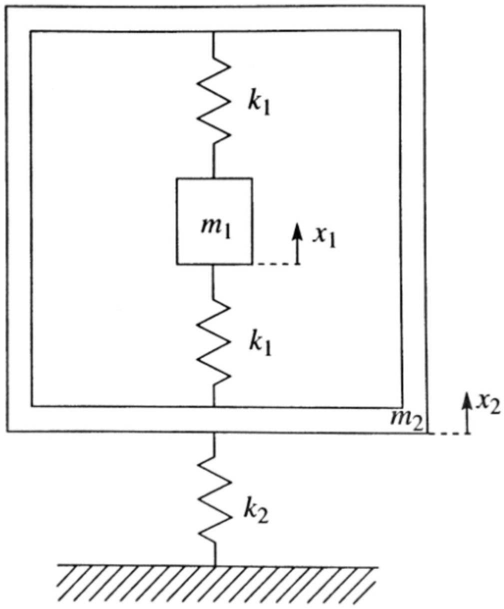

Figure P4.1

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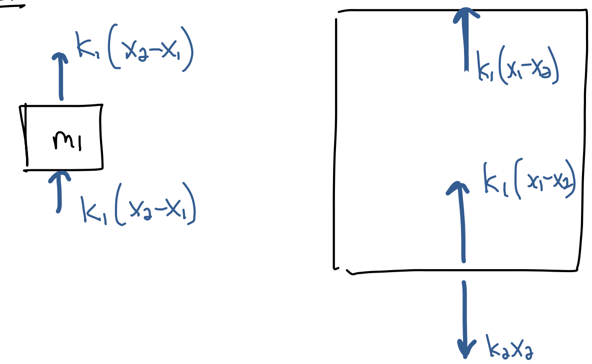


x_1-equation

$$m_1\ddot{x}_1 = k_1(x_2-x_1) + k_1(x_2-x_1)$$

$$m_1\ddot{x}_1 + 2k_1x_1 - 2k_1x_2 = 0$$

x_2-equation

$$m_2\ddot{x}_2 = k_1(x_1-x_2) + k_1(x_1-x_2) - k_2x_2$$

$$m_2\ddot{x}_2 - 2kx_1 + (2k_1+k_2)x_2 = 0$$

Put into matrix form

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M}\begin{vmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{vmatrix} + \underbrace{\begin{bmatrix} 2k_1 & -2k_1 \\ -2k_1 & 2k_1+k_2 \end{bmatrix}}_{K}\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

To find the eigenvalues, solve

$$\det(K - \omega^2 M) = 0$$

$$\det\left(\begin{vmatrix} 2k_1-\omega^2 m_1 & -2k_1 \\ -2k_1 & 2k_1+k_2-\omega^2 m_2 \end{vmatrix}\right) = 0$$

$$(2k_1-\omega^2 m_1)((2k_1+k_2)-\omega^2 m_2) - (2k_1)^2 = 0$$

$$2k_1(2k_1+k_2) - 2k_1\omega^2 m_2 - (2k_1+k_2)\omega^2 m_1 + \omega^4 m_1 m_2 - (2k_1)^2 = 0$$

$$m_1 m_2 \omega^4 - 2k_1 m_2 \omega^2 - (2k_1+k_2)m_1\omega^2 + 2k_1k_2 = 0$$

$$m_1 m_2 \omega^4 - \left(2k_1 m_2 + (2k_1+k_2)m_1\right)\omega^2 + 2k_1k_2 = 0$$

Problem 4.1 (cont.)

$$m_1 m_2 \omega^4 - \left(2k_1 m_2 + (2k_1+k_2)m_1\right)\omega^2 + 2k_1k_2 = 0$$ This is quadratic in ω^2

$$\omega^2 = \frac{\left(2k_1m_2 + (2k_1+k_2)m_1\right) \pm \sqrt{\left(2k_1m_2+(2k_1+k_2)m_1\right)^2 - 8m_1m_2k_1k_2}}{2m_1m_2}$$

we find

$$\omega_1^2 \cong 2.72\times10^5 \longrightarrow \boxed{\omega_1 \cong 521 \tfrac{rad}{s}}$$

$$\omega_2^2 \cong 6.63\times10^6 \longrightarrow \boxed{\omega_2 = 2574.6 \tfrac{rad}{s}}$$

Now, plug ω_1^2 and ω_2^2 into

$[K-\omega_i^2 M]\bar{X}=0$ and solve for X

for ω_1

$$\begin{vmatrix} 2k_1-\omega_1^2m_1 & -2k_1 \\ -2k_1 & 2k_1+k_2-\omega_1^2m_2 \end{vmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix} = 0$$

$$(2k_1-\omega_1^2m_1)\bar{X}_1 - 2k_1\bar{X}_2 = 0 \longrightarrow 5728.4\bar{x}_1 - 6000\bar{x}_2 = 0$$

$$-2k_1\bar{X}_1 + (2k_1+k_2-\omega_1^2m_2)\bar{X}_2 = 0 \rightarrow \underbrace{-6000\bar{x}_1 + 6284.4\bar{x}_2 = 0}$$

set $X_1 = 1$ then

$$X_2 = \frac{5728.4}{6000}\bar{X}_1 = 0.954$$

Notice that these two equations are the same, as we expect because we chose ω_1^2 to make $\det(\cdot)=0$

so $\boxed{\bar{X}_1 = \begin{vmatrix}1\\0.954\end{vmatrix}}$

for ω_2

$$\begin{vmatrix} 2k_1-\omega_2^2m_1 & -2k_1 \\ -2k_1 & 2k_1+k_2-\omega_2^2m_2 \end{vmatrix}\begin{vmatrix}\bar{X}_1\\ \bar{X}_2\end{vmatrix} = 0$$

$$(2k_1-\omega_1^2m_1)\bar{X}_1 - 2k_1\bar{X}_2 = 0 \longrightarrow -628.44\bar{x}_1 - 6000\bar{x}_2 = 0$$

$$-2k_1\bar{X}_1 + (2k_1+k_2-\omega_1^2m_2)\bar{X}_2 = 0 \rightarrow -6000\bar{x}_1 + (-57284)\bar{x}_2 = 0$$

pick $\bar{X}_1 = 1$

$$\bar{X}_2 = \frac{-628.44}{6000}\bar{X}_1 = -0.105$$

so $\boxed{\bar{X}_2 = \begin{vmatrix}1\\-0.105\end{vmatrix}}$

Problem 4.2

4.2. Find the three natural frequencies and their associated eigenvectors for the system illustrated in Figure P4.2. $m_1 = .5$ kg, $m_2 = .5$ kg, $m_3 = .02$ kg, $k_1 = 1 \times 10^3$ N/m, $k_2 = 1.5 \times 10^3$ N/m, $k_3 = .2 \times 10^3$ N/m.

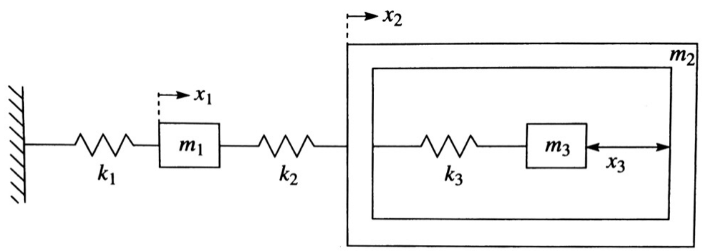
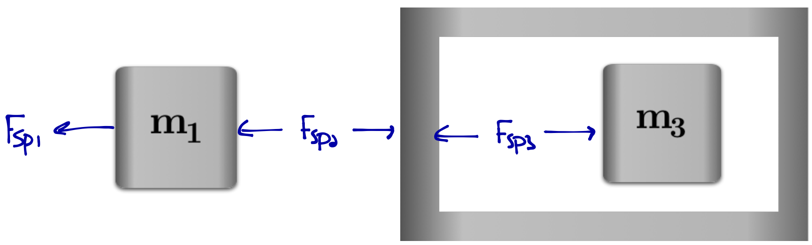


Figure P4.2



$F_{sp1} = k_1 x_1$

$F_{sp2} = k_2(x_1 - x_2)$

$F_{sp3} = k_3 x_3$ ← x_3 is the relative motion of m_3

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) \longrightarrow m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3x_3 \longrightarrow m_2\ddot{x}_2 - k_2x_1 + k_2x_2 + k_3x_3 = 0$$

$$m_3(\underbrace{\ddot{x}_2 + \ddot{x}_3}) = -k_3x_3 \longrightarrow m_3(\ddot{x}_2 + \ddot{x}_3) + k_3x_3 = 0$$

Must write the total accel of m_3

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & m_3 & m_3 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2 & k_3 \\ 0 & 0 & k_3 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{0}$$

- Solve $\det(K - \omega_i^2 M) = 0$ for eigenvalues ω_i^2
- For each ω_i^2, solve $[K - \omega_i^2 M]X_i = 0$ for eigenvector X_i

Find:

$\omega_1^2 = 835.72 \left(\frac{rad}{s}\right)^2$ $\quad X_1 = \begin{bmatrix} 0.625 \\ 1.145 \\ 0.104 \end{bmatrix}$

$\omega_2^2 = 6958.51 \left(\frac{rad}{s}\right)^2$ $\quad X_2 = \begin{bmatrix} 1.12 \\ -0.731 \\ -1.673 \end{bmatrix}$

$\omega_3^2 = 10747.44 \left(\frac{rad}{s}\right)^2$ $\quad X_3 = \begin{bmatrix} -0.255 \\ 0.488 \\ -7.019 \end{bmatrix}$

Problem 4.8

4.8. Find the eigenvectors and natural frequencies for the system illustrated in Figure P4.8. Comment on the physical behavior. $m_1 = 2$ kg, $m_2 = .02$ kg, $m_3 = 2$ kg, $k_1 = 1000$ N/m, $k_2 = 20$ N/m, $k_3 = 2000$ N/m, $k_4 = 20$ N/m, $k_5 = 1000$ N/m.

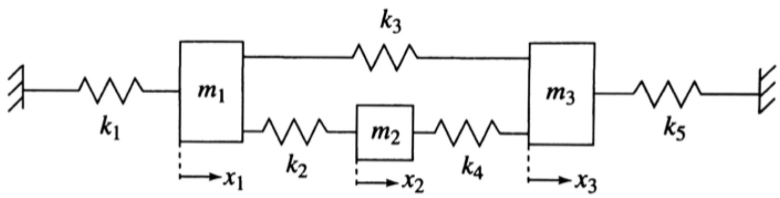


Figure P4.8

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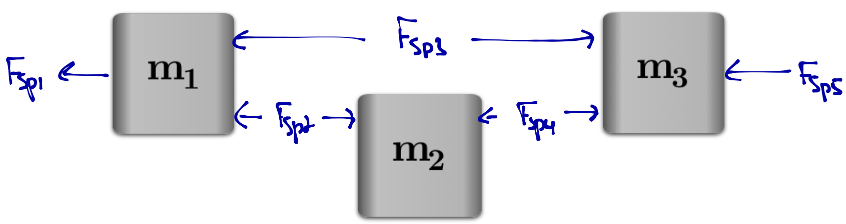


$$m_1\ddot{x}_1 = -F_{sp1} - F_{sp2} - F_{sp3} = -kx_1 - k_2(x_1 - x_2) - k_3(x_1 - x_3)$$

$$m_2\ddot{x}_2 = F_{sp2} - F_{sp4} = k_2(x_1 - x_2) - k_4(x_2 - x_3)$$

$$m_3\ddot{x}_3 = F_{sp3} + F_{sp4} - F_{sp5} = k_3(x_1 - x_3) + k_4(x_2 - x_3) - k_5 x_3$$

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} k_1 + k_2 + k_3 & -k_2 & -k_3 \\ -k_2 & k_2 + k_4 & -k_4 \\ -k_3 & -k_4 & k_3 + k_4 + k_5 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$M\ddot{X} + KX = 0$$

To solve the eigenvalue problem, solve:

$$\det\left([K - \omega^2 M]\right) = 0$$

Then, for each eigenvalue, ω_i^2, solve:

$$[K - \omega_i^2 M]\bar{X}_i = 0 \quad \text{for } \bar{X}_i$$

to find the eigenvectors.

See the IPython Notebook for these solutions.

Problem 4.8 (cont.)

Looking at the physical meaning of these:

X1 - x1 and x3 move together, with x2 moving in phase with them but at a higher amplitude. This makes sense given the symmetry of the problem.

X2 - x1 and x3 move at small amplitude together, the small mass, m2, moves more than the heavier m1 and m2. This again makes sense.

X3 - x1 and x3 move opposite of one another. Given the symmetry, the exert balancing forces on m2, meaning x2 is stationary.

Problem 4.8 (cont.)

Let's also write these equations of motion in <u>state-space form</u>

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2+k_3 & -k_2 & -k_3 \\ -k_2 & k_2+k_4 & -k_4 \\ -k_3 & -k_4 & k_3+k_4+k_5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\ddot{x}_1 = \frac{1}{m_1}\left[-kx_1 - k_2(x_1-x_2) - k_3(x_1-x_3)\right]$$

$$\ddot{x}_2 = \frac{1}{m_2}\left[k_2(x_1-x_2) - k_4(x_2-x_3)\right]$$

$$\ddot{x}_3 = \frac{1}{m_3}\left[k_3(x_1-x_3) + k_4(x_2-x_3) - k_5x_3\right]$$

Define the state vector $\bar{\omega} = [x_1\ \dot{x}_1\ x_2\ \dot{x}_2\ x_3\ \dot{x}_3]^T$ ← Remember that the state vector is usually written as $\bar{x}$, we just use $\bar{\omega}$ to avoid confusion with our "x" generalized coords.

We want to write

$$\dot{\bar{\omega}} = A\bar{\omega} + Bu$$

$$\dot{\bar{\omega}} = \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \\ \dot{\omega}_4 \\ \dot{\omega}_5 \\ \dot{\omega}_6 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \\ \dot{x}_3 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \frac{1}{m_1}\left[-kx_1 - k_2(x_1-x_2) - k_3(x_1-x_3)\right] \\ \dot{x}_2 \\ \frac{1}{m_2}\left[k_2(x_1-x_2) - k_4(x_2-x_3)\right] \\ \dot{x}_3 \\ \frac{1}{m_3}\left[k_3(x_1-x_3) + k_4(x_2-x_3) - k_5x_3\right] \end{bmatrix}$$

Now write this in terms of the states, $\bar{\omega}$, an put into matrix form

$$\bar{\omega} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-(k_1+k_2+k_3)}{m_1} & 0 & \frac{k_2}{m_1} & 0 & \frac{k_3}{m_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & 0 & \frac{-(k_2+k_4)}{m_2} & 0 & \frac{k_4}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{k_3}{m_3} & 0 & \frac{k_4}{m_3} & 0 & \frac{-(k_3+k_4+k_5)}{m_3} & 0 \end{bmatrix}}_{A} \bar{\omega}$$

<u>$B = 0$</u>

Problem 4.9

4.9. Figure P4.9 shows a double pendulum system, which also can be looked at as a model of a two-link robotic manipulator. Find the equations of motion about the system's stable equilibrium position ($\theta_1 = \theta_2 = 0$). Once you've found them, linearize the equations by assuming that all the angular deflections are small. Then calculate the system's natural frequencies and eigenvectors for the given parameter values. $m_1 = 2$ kg, $m_2 = 2$ kg, $l_1 = 1$ m, $l_2 = 1.5$ m, $g = 9.81$ m/s^2.

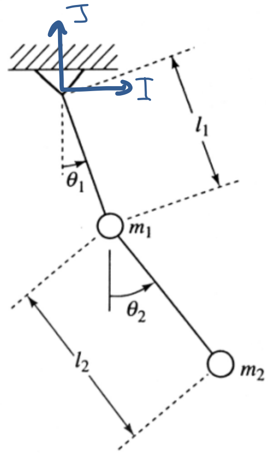


Figure P4.9

Use Lagrange's method

$$\bar r_{m1} = l_1\sin\theta_1\bar I - l_1\cos\theta_1\bar J$$

$$\dot{\bar r}_{m1} = l_1\dot\theta\cos\theta\bar I + l_1\dot\theta_1\sin\theta_1\bar J$$

$$\bar r_{m2} = (l_1\sin\theta_1 + l_2\sin\theta_2)\bar I - (l_1\cos\theta_1 + l_2\cos\theta_2)\bar J$$

$$\dot{\bar r}_{m2} = (l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2)\bar I + (+l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\sin\theta_2)\bar J$$

If we are looking at motion around $\theta_1=0, \theta_2=0$, then almost all velocity is in the I direction. We can just use that component.

$$\dot{\bar r}_{m1} = l_1\dot\theta_1\cos\theta_1 I \quad \text{and} \quad \dot{\bar r}_{m2} = (l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2) I$$

We're also assuming small angles so $\cos\theta \approx 1$, so

$$T = \frac{1}{2}m_1(l_1\dot\theta_1)^2 + \frac{1}{2}m_2(l_1\dot\theta_1 + l_2\dot\theta_2)^2$$

$$= \frac{1}{2}m_1 l_1^2\dot\theta_1^2 + \frac{1}{2}m_2(l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\dot\theta_1\dot\theta_2)$$

$$T = \frac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2 + \frac{1}{2}m_2l_2^2\dot\theta_2^2 + \frac{1}{2}m_2l_1l_2\dot\theta_1\dot\theta_2$$

$$V = -m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$L = T - V = \frac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2 + \frac{1}{2}m_2l_2^2\dot\theta_2^2 + \frac{1}{2}m_2l_1l_2\dot\theta_1\dot\theta_2 + m_1gl_1\cos\theta_1 + m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

Problem 4.9 (cont.)

$$L = T - V = \frac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2 + \frac{1}{2}m_2 l_2^2\dot\theta_2^2 + \frac{1}{2}m_2 l_1 l_2\dot\theta_1\dot\theta_2 + m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$\underline{\theta_1}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_1}\right) - \frac{\partial L}{\partial\theta_1} = 0$$

$$\frac{\partial L}{\partial\dot\theta_1} = (m_1+m_2)l_1^2\dot\theta_1 + \frac{1}{2}m_2 l_1 l_2\dot\theta_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_1}\right) = (m_1+m_2)l_1^2\ddot\theta_1 + \frac{1}{2}m_2 l_1 l_2\ddot\theta_2$$

$$\frac{\partial L}{\partial\theta_1} = -m_1 g l_1\sin\theta_1 - m_2 g l_1\sin\theta_1 \leftarrow \text{small angles} \rightarrow -m_1 g l_1\theta_1 - m_2 g l_1\theta_1 = -(m_1+m_2)g l_1\theta_1$$

$$(m_1+m_2)l_1^2\ddot\theta_1 + \frac{1}{2}m_2 l_1 l_2\ddot\theta_2 + (m_1+m_2)g l_1\theta_1 = 0$$

$\underline{\theta_2 \text{ equation}}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right) - \frac{\partial L}{\partial\theta_2} = 0$$

$$\frac{\partial L}{\partial\dot\theta_2} = m_2 l_2^2\dot\theta_2 + \frac{1}{2}m_2 l_1 l_2\dot\theta_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right) = m_2 l_2^2\ddot\theta_2 + \frac{1}{2}m_2 l_1 l_2\ddot\theta_1$$

$$\frac{\partial L}{\partial\theta_2} = -m_2 g l_2\sin\theta_2 \leftarrow \text{small angles} \rightarrow -m_2 g l_2\theta_2$$

$$m_2 l_2^2\ddot\theta_2 + \frac{1}{2}m_2 l_1 l_2\ddot\theta_1 + m_2 g l_2\theta_2 = 0$$

In matrix form:

$$\underbrace{\begin{bmatrix} (m_1+m_2)l_1^2 & \frac{1}{2}m_2 l_1 l_2 \\ \frac{1}{2}m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix}}_{M}\begin{vmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{vmatrix} + \underbrace{\begin{vmatrix} (m_1+m_2)g l_1 & 0 \\ 0 & m_2 g l_2 \end{vmatrix}}_{K}\begin{vmatrix}\theta_1 \\ \theta_2\end{vmatrix} = \begin{vmatrix}0 \\ 0\end{vmatrix}$$

now, just follow our standard eigenvalue/eigenvector solution procedure

solve $\det(K - \omega^2 M) = 0$ for ω^2 and using ω_i solve $(K - \omega_i^2 M)(\bar{X}_i) = 0$

we find
$$\boxed{\begin{matrix} \omega_1 = 2.37\ \frac{rad}{s} & \bar{X}_1 = \begin{bmatrix} 0.19 \\ 0.38 \end{bmatrix} \\ \omega_2 = 3.62\ \frac{rad}{s} & X_2 = \begin{bmatrix} -1/2 \\ 1/3 \end{bmatrix} \end{matrix}}$$

Problem 4.12

4.12. Find the equations of motion, linearize them, and find the natural frequencies and eigenvectors for the system illustrated in Figure P4.12. $m_1 = 2$ kg, $m_2 = 20$ kg, $m_3 = 1$ kg, $k_1 = 1000$ N/m, $l = 1$ m.

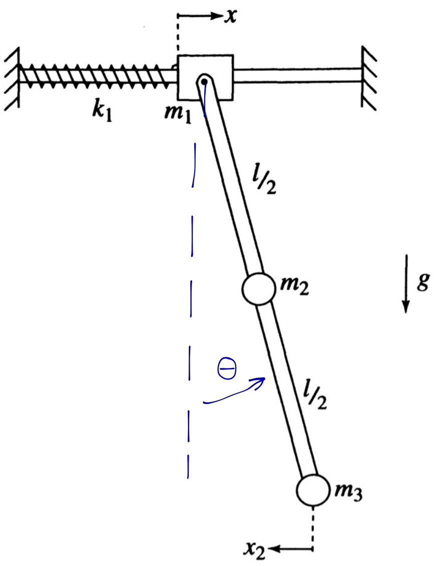
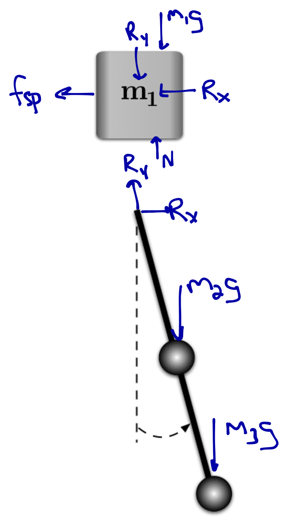


Figure P4.12



Find eq. of motion using Lagrange's Method

This system has 2DOF, choose $\bar{q} = (x, \theta)$

$$T = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \left(\dot{x} + \tfrac{l}{2}\dot{\theta}\right) + \tfrac{1}{2} m_3 \left(\dot{x} + l\dot{\theta}\right)$$

$$V = \tfrac{1}{2} k x^2 - m_2 g \tfrac{l}{2} \cos\theta - m_3 g l \cos\theta$$

We are assuming small angles here.

Aside: To write the velocities of m_2 and m_3, write their position, take time deriv, and linearize.

$$L = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \left(\dot{x} + \tfrac{l}{2}\dot{\theta}\right)^2 + \tfrac{1}{2} m_3 \left(\dot{x} + l\dot{\theta}\right)^2 - \left[\tfrac{1}{2} k x^2 - m_2 g \tfrac{l}{2} \cos\theta - m_3 g l \cos\theta\right]$$

<u>for x</u>

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2\left(\dot{x} + \tfrac{l}{2}\dot{\theta}\right) + m_3\left(\dot{x} + l\dot{\theta}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1 \ddot{x} + m_2\left(\ddot{x} + \tfrac{l}{2}\ddot{\theta}\right) + m_3\left(\ddot{x} + l\ddot{\theta}\right) = (m_1 + m_2 + m_3)\ddot{x} + \left(m_2 \tfrac{l}{2} + m_3 l\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$(m_1 + m_2 + m_3)\ddot{x} + \left(m_2 \tfrac{l}{2} + m_3 l\right)\ddot{\theta} + kx = 0$$

<u>for θ</u>

$$\frac{\partial L}{\partial \dot{\theta}} = m_2\left(\dot{x} + \tfrac{l}{2}\dot{\theta}\right)\tfrac{l}{2} + m_3\left(\dot{x} + l\dot{\theta}\right) l$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2 \tfrac{l}{2} \ddot{x} + m_2\left(\tfrac{l}{2}\right)^2 \ddot{\theta} + m_3 l \ddot{x} + m_3 l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -m_2 g \tfrac{l}{2} \sin\theta - m_3 g l \sin\theta$$

$$\left(m_2 \tfrac{l}{2} + m_3 l\right)\ddot{x} + \left(m_2\left(\tfrac{l}{2}\right)^2 + m_3 l^2\right)\ddot{\theta} + \left(m_2 \tfrac{l}{2} + m_3 l\right) g \sin\theta = 0$$

linearize $\sin\theta \approx \theta$

$$\left(m_2 \tfrac{l}{2} + m_3 l\right)\ddot{x} + \left(m_2\left(\tfrac{l}{2}\right)^2 + m_3 l^2\right)\ddot{\theta} + \left(m_2 \tfrac{l}{2} + m_3 l\right) g \theta = 0$$

Problem 4.12 (cont.)

In matrix form:

$$\underbrace{\begin{bmatrix} m_1+m_2+m_3 & m_2\frac{l}{2}+m_3 l \\ m_2\frac{l}{2}+m_3 l & m_2\left(\frac{l}{2}\right)^2+m_3 l^2 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}}+\underbrace{\begin{bmatrix} k & 0 \\ 0 & \left(m_2\frac{l}{2}+m_3 l\right)g \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{X}=\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

Now, solve:

1) $\det\left(K-\omega_i^2 M\right)=0$ for eigenvalues ω_i^2

2) For each ω_i^2, solve $\left[K-\omega_i^2 M\right]X_i=0$ for eigenvector X_i

Problem 4.13

4.13. Consider the system illustrated in Figure P4.13. The entire mass is concentrated in three places; the rest of the rigid bar is massless. Find the system's equations of motion. Determine the eigenvectors and natural frequencies. $m_1 = 1$ kg, $m_2 = 1$ kg, $m_3 = 1$ kg, $k = 2$ N/m, $l = 1$ m.

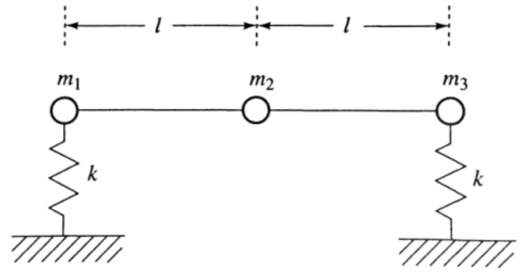
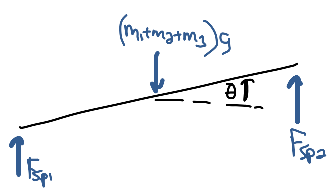


Figure P4.13

FBD



Because all three masses and equally spaced, the COM is at m_2. So, we can treat gravity just acting at that point.

Use coordinates x (vertical motion of COM) and θ (rotation of bar)

X-equation

$$(m_1+m_2+m_3)\ddot{x} = -k(x+l\theta) - k(x-l\theta) = 0$$

$$(m_1+m_2+m_3)\ddot{x} + 2kx = 0$$

because of the symmetry, we can also write x about the equil. and therefore it doesn't show up in the equations of motion

θ- equation

$\Sigma M = I\ddot{\theta}$ where I is the moment of inertia about the COM

$$I\ddot{\theta} = -k(x+l\theta)l + k(x-l\theta)l = 0$$

$$I\ddot{\theta} + 2kl^2\theta = 0$$

$$I = (m_1 l^2 + m_3 l^2)$$

In matrix form:

$$\underbrace{\begin{bmatrix} m_1+m_2+m_3 & 0 \\ 0 & (m_1+m_3)l^2 \end{bmatrix}}_{M} \begin{vmatrix} \ddot{x} \\ \ddot{\theta} \end{vmatrix} + \underbrace{\begin{bmatrix} 2k & 0 \\ 0 & 2kl^2 \end{bmatrix}}_{K} \begin{vmatrix} x \\ \theta \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

Problem 4.13 (cont.)

$$\underbrace{\begin{bmatrix} m_1+m_2+m_3 & 0 \\ 0 & (m_1+m_3)\ell^2 \end{bmatrix}}_{M}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \underbrace{\begin{bmatrix} 2k & 0 \\ 0 & 2k\ell^2 \end{bmatrix}}_{K}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The eigenvalue problem is then:

$$\det\left(K-\omega^2 M\right)=0 \longrightarrow \det\left(\begin{bmatrix} 2k-\omega^2(m_1+m_2+m_3) & 0 \\ 0 & 2k\ell^2-\omega^2(m_1+m_3)\ell^2 \end{bmatrix}\right)=0$$

find $\omega_1^2 = \frac{4}{3}$ and $\omega_2^2 = 2$

$$\boxed{\omega_1 = 1.155 \qquad \omega_2 = 1.414}$$

To find the eigenvectors solve $[K-\omega_i^2 M]\bar{X}_i=0$

for $\omega_1^2 = \frac{4}{3}$

$$\left[K-\tfrac{4}{3}M\right]\bar{X}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 4/3 \end{bmatrix}\begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = 0 \longrightarrow \boxed{\bar{X}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}}$$

for $\omega_2^2 = 2$

$$\left[K-2M\right]\bar{X}_2 = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow \boxed{\bar{X}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}}$$

As we expect from our uncoupled equations of motion our mode shapes are uncoupled as well.