MCHE 485: Mechanical Vibrations
Spring 2019 – Homework 3

Assigned:	Tuesday, March 12th
Due:	Friday, March 22nd, 5pm

Assignment: From "Principles of Vibration" by Benson Tongue, problems:
2.79, 2.80, 2.100, 3.18

Set up the integrals in the solution such that they could be given to a Calculus student to solve. Be sure to define all of the terms needed:
3.4, 3.8, 3.9, 3.15

Submission: Emailed *single* pdf document:

- to joshua.vaughan@louisiana.edu
- with subject line and filename `ULID-MCHE485-HW3`, where `ULID` is your ULID
- *Note:* Submissions with incorrect filenames or submitted as multiple images/pdfs will be rejected.

Problem 2.79

2.79. The peak amplitude response $|g(\omega_p)|$ is equal to 5 cm for a direct force excited (Figure P2.79), damped SDOF system. If you know that $m = .03$ kg, $c = .048$ N·s/m, and $k = 12$ N/m, can you determine $|g(0)|$?

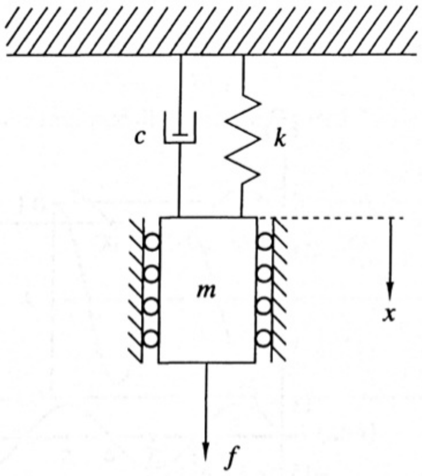


Figure P2.79

We know that:

$$\frac{|G(\omega_p)|}{|G(\omega_0)|} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \left(\text{and} \approx \frac{1}{2\zeta}\right) \longleftarrow \text{Use this to estimate } |G(\omega_0)|$$

Here:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{0.03}} = 20\ \frac{\text{rad}}{\text{s}}$$

$$2\zeta\omega_n = \frac{c}{m} \longrightarrow \zeta = \frac{c}{2m\omega_n} = 0.04$$

So,

$$\frac{5\text{cm}}{|G(0)|} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

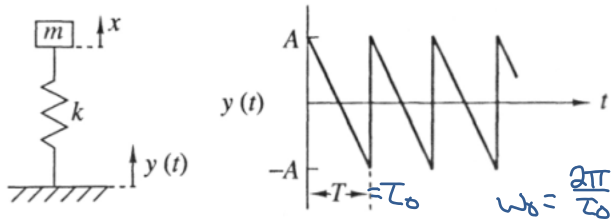
$$\frac{5}{|G(0)|} = 12.51$$

$$G(0) = \frac{5}{12.51} = 0.399 \simeq 0.4 \quad \left(\text{If you just use the } \frac{1}{2\zeta} \text{ approximation, you get } 0.4\right)$$

Problem 3.8

3.8. For the system shown in Figure P3.8, $m = 20$ kg and $k = 400$ N/m. The base of the system is given a vertical displacement $y(t)$ in the form of a sawtooth function with amplitude $A = 3$ cm and period $T = 2$ seconds. Compute the response of the system and identify the dominant response harmonic.

$$m\ddot{x} + kx = ky$$
$$\ddot{x} + \frac{k}{m}x = \frac{k}{m}y$$
$$\ddot{x} + 20x = 20y$$



$$\omega_0 = \frac{2\pi}{\tau_0} = \frac{2\pi}{T} \longrightarrow \frac{2\pi}{(2s)} = \pi \ \frac{rad}{s}$$

Figure P3.8

Use Fourier analysis to describe sawtooth signal in terms of sin and cos

slope of sawtooth - $\dfrac{-A-A}{T-0} = \dfrac{-2A}{T}$ $\qquad y(t) = \dfrac{-2A}{T}t + A \ , \ 0 < t < T$

$$a_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\cos(n\omega_0 t)\,dt \qquad a_0 = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\,dt$$

$$b_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)\,dt$$

$$f(t) = \sum_{n=0}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

here, $f(t) = y(t)$

$$a_0 = \frac{\omega_0}{\pi}\int_0^T \left(\frac{-2A}{T}t + A\right)dt = \frac{A\omega_0}{\pi}\int_0^T \left(\frac{-2}{T}t + 1\right)dt = \frac{A\omega_0}{\pi}\left[\frac{-1}{T}t^2 + t\right]\Big|_0^T = \frac{A\omega_0}{\pi}(-T+T) = 0$$

$$a_n = \frac{\omega_0}{\pi}\int_0^T A\underbrace{\left(\frac{-2}{T}t + 1\right)}_{u}\underbrace{\cos(n\omega_0 t)\,dt}_{dv}$$

Integrate by parts $\int u\,dv = uv - \int v\,du$

$$du = \frac{-2}{T}dt \qquad v = \int \cos(n\omega_0 t)\,dt = \frac{1}{n\omega_0}\sin(n\omega_0 t)$$

$$a_n = \frac{A\omega_0}{\pi}\left[\underbrace{\left(\frac{-2}{T}t + 1\right)}_{u}\underbrace{\left(\frac{1}{n\omega_0}\sin(n\omega_0 t)\right)}_{v}\Big|_0^T - \int_0^T \underbrace{\left[\frac{1}{n\omega_0}\sin(n\omega_0 t)\right]}_{v}\underbrace{\left[\frac{-2}{T}dt\right]}_{du}\right]$$

$$a_n = \frac{A\omega_0}{\pi}\left[\frac{-1}{n\omega_0}\sin(n\omega_0 T)\right] + \frac{2}{n\omega_0 T}\left[\frac{-1}{n\omega_0}\cos(n\omega_0 t)\right]\Big|_0^T$$

$$a_n = \frac{A\omega_0}{\pi}\left[\cancel{\frac{-1}{n\omega_0}\sin(2n\pi)} - \frac{2}{(n\omega_0)^2}\left(\frac{\omega_0}{2\pi}\right)\cancel{\left[\cos(2n\pi) - \cos(0)\right]}\right] \to 0$$

$\sin(2n\pi) = 0 \qquad \cos(2n\pi) = 1 \qquad \cos(0) = 1$

$$a_n = 0$$

Problem 3.8 (cont.)

$$b_n = \frac{\omega_0}{\pi}\int_0^{\frac{2\pi}{\omega_0}} f(t)\sin(n\omega_0 t)\,dt \qquad f(t) = y(t) = A\left(\frac{-2}{T}t + 1\right)$$

$$b_n = \frac{A\omega_0}{\pi}\int_0^T \underbrace{\left(\frac{-2}{T}t + 1\right)}_{u}\underbrace{\sin(n\omega_0 t)\,dt}_{dv} \longleftarrow \text{integrate by parts}$$

$$du = \frac{-2}{T}dt \qquad v = \int \sin(n\omega_0 t)\,dt = \frac{-1}{n\omega_0}\cos(n\omega_0 t)$$

$$b_n = \frac{A\omega_0}{\pi}\left[\left(\frac{-2}{T}t + 1\right)\left(\frac{-1}{n\omega_0}\cos(n\omega_0 t)\right)\Bigg|_0^T - \int_0^T \frac{-1}{n\omega_0}\cos(n\omega_0 t)\left(\frac{-2}{T}\,dt\right)\right]$$

$$= \frac{A\omega_0}{\pi}\left[(-1)\left(\frac{-1}{n\omega_0}\cos(2n\pi)\right) - \left(\frac{-1}{n\omega_0}\cos(0)\right) - \underbrace{\cancel{\frac{2}{nT\omega_0}\left[\frac{1}{n\omega_0}\sin(n\omega_0 t)\Big|_0^T\right]}}_{0}\right]$$

$\sin(2n\pi) = 0$

$\sin(0) = 0$

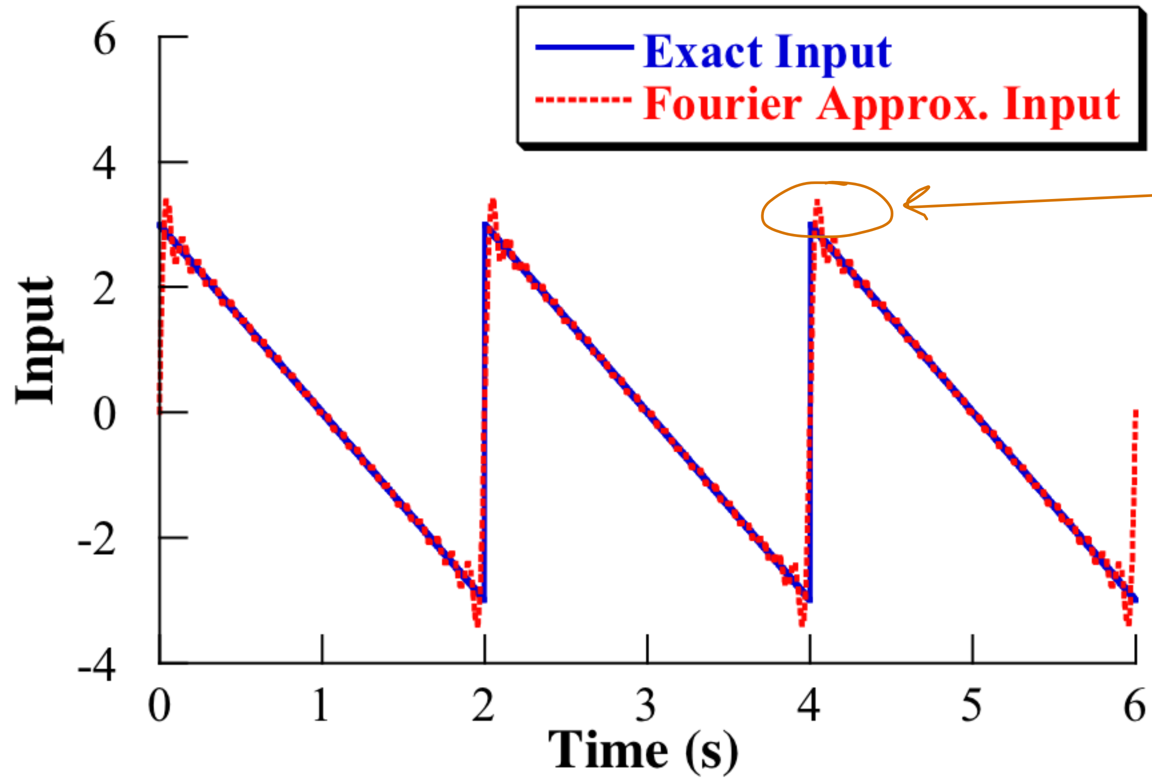
$$= \frac{A\omega_0}{\pi}\left[\frac{1}{n\omega_0} + \frac{1}{n\omega_0}\right] = \frac{2A}{n\pi}$$

So $$y(t) = \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\omega_0 t)$$

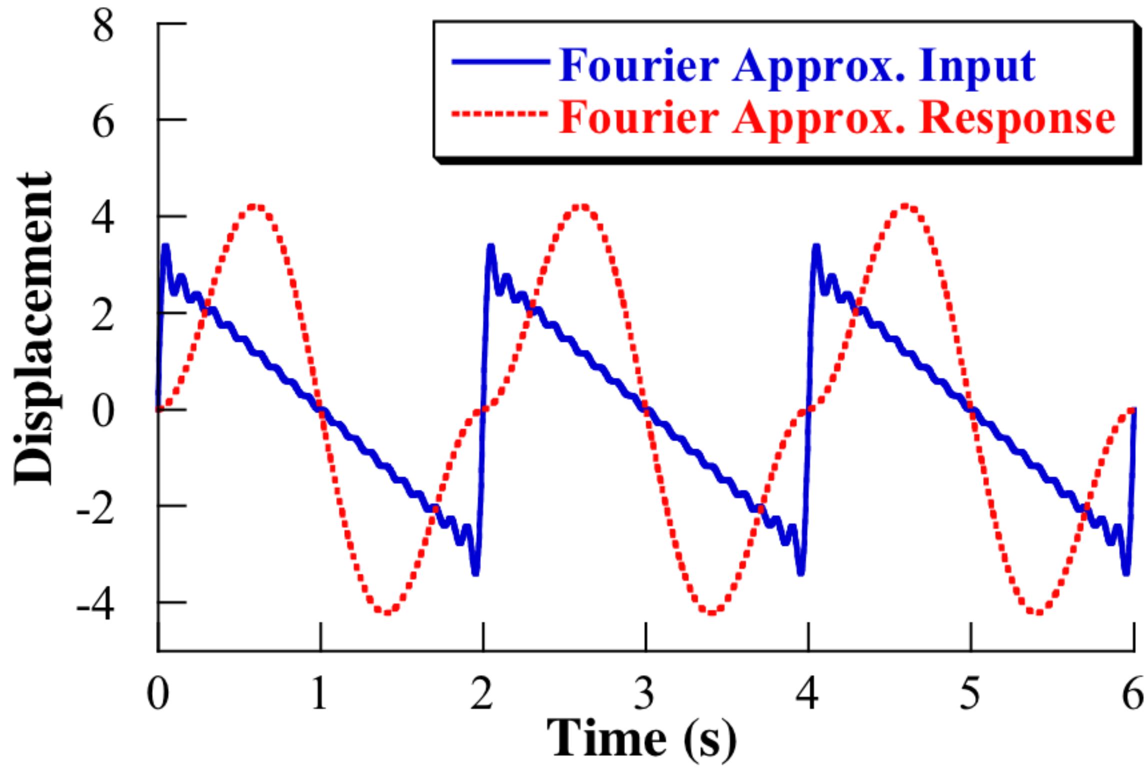
and
$$x(t) = \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{\omega_n^2}{\omega_n^2 - (n\omega_0)^2}\,\frac{1}{n}\sin(n\omega_0 t)$$

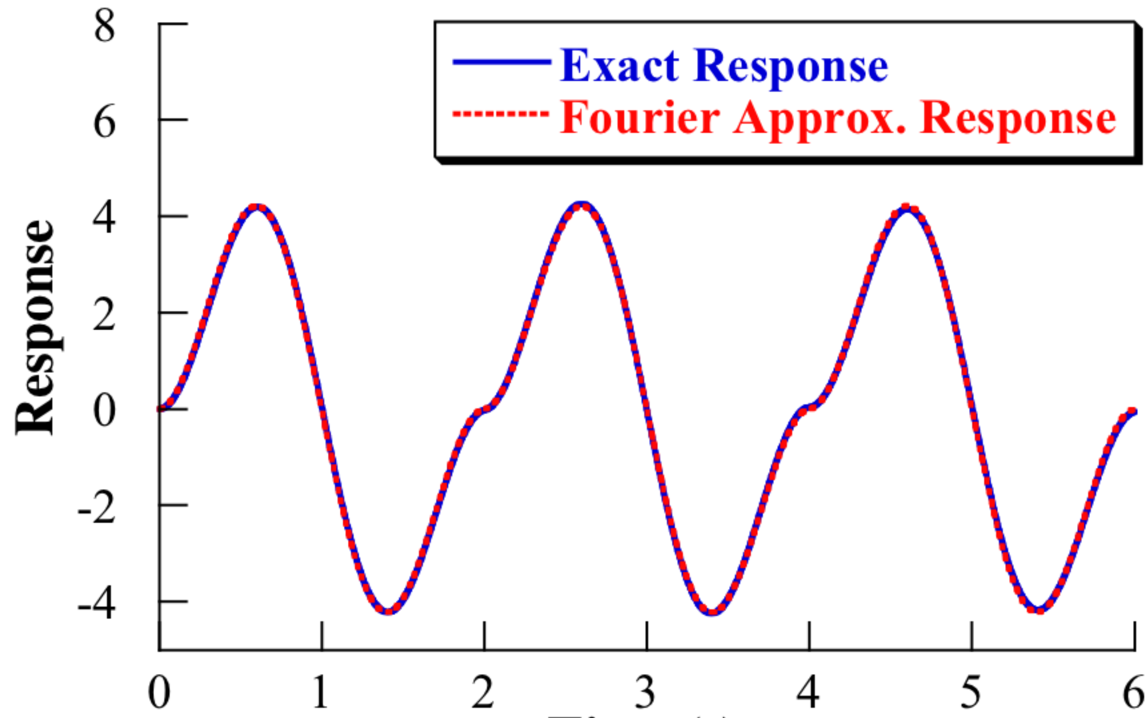
Problem 3.8 (cont.)

20-term approximation

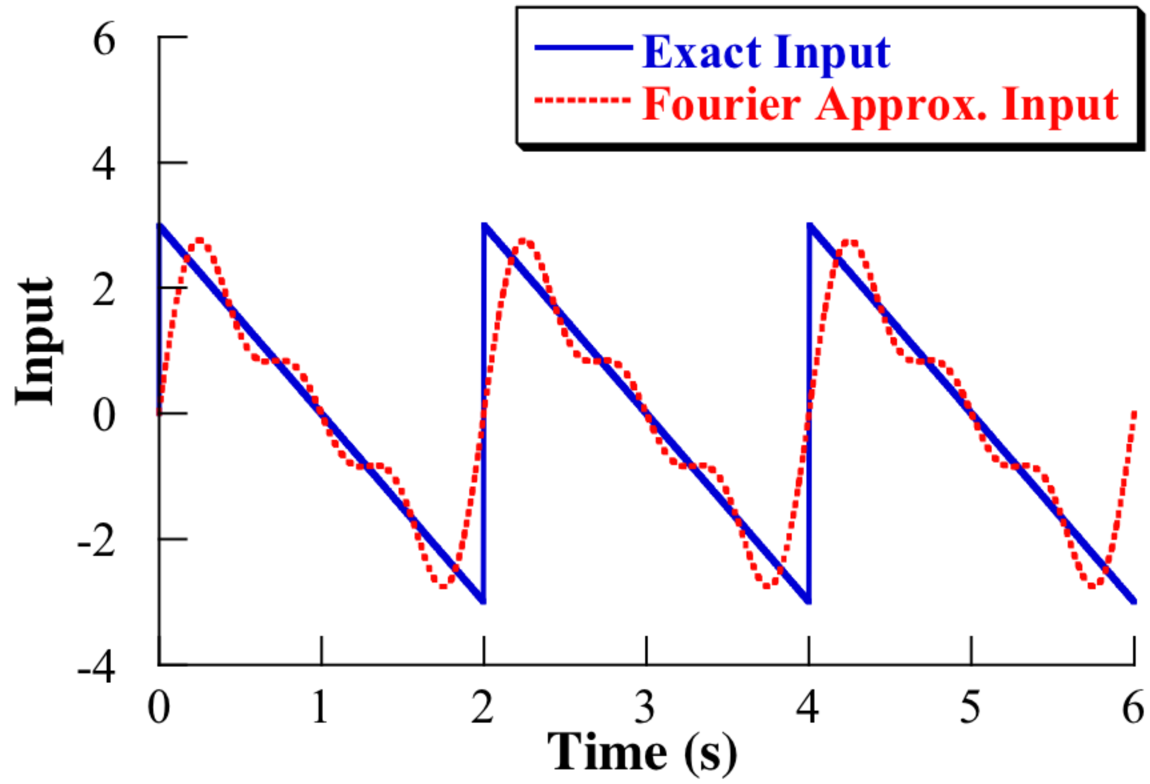


This overshoot never goes away. This is called the Gibbs phenomenon.

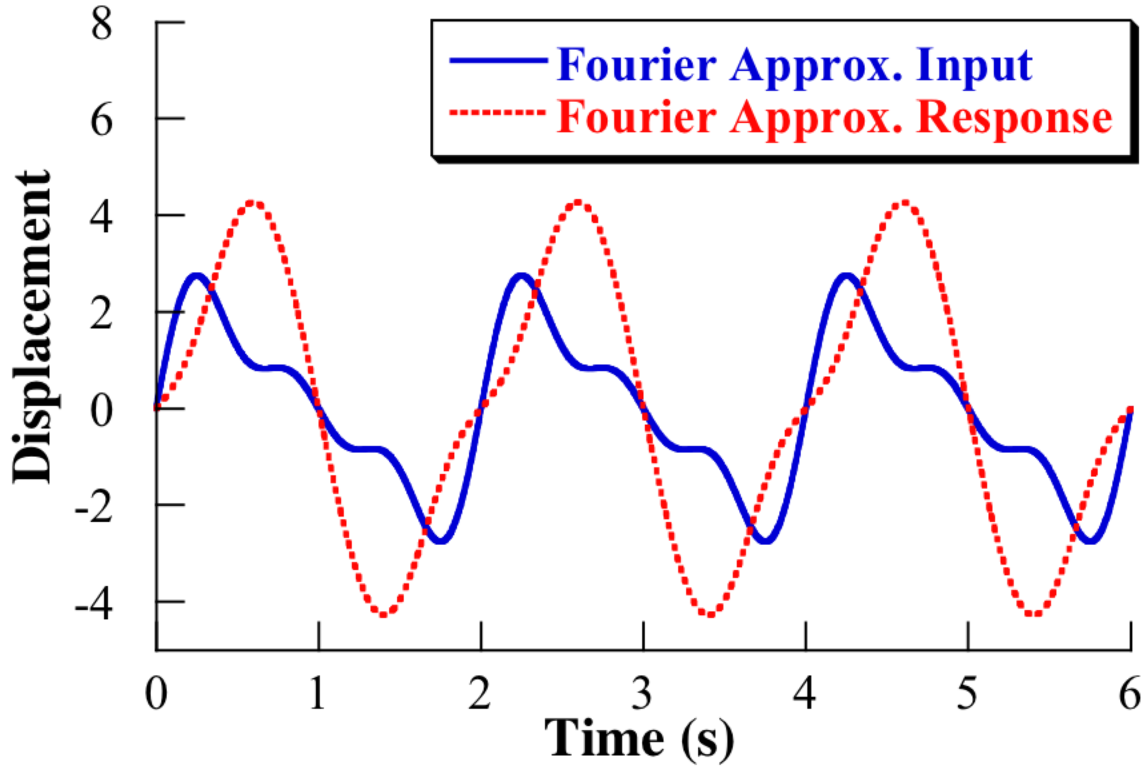


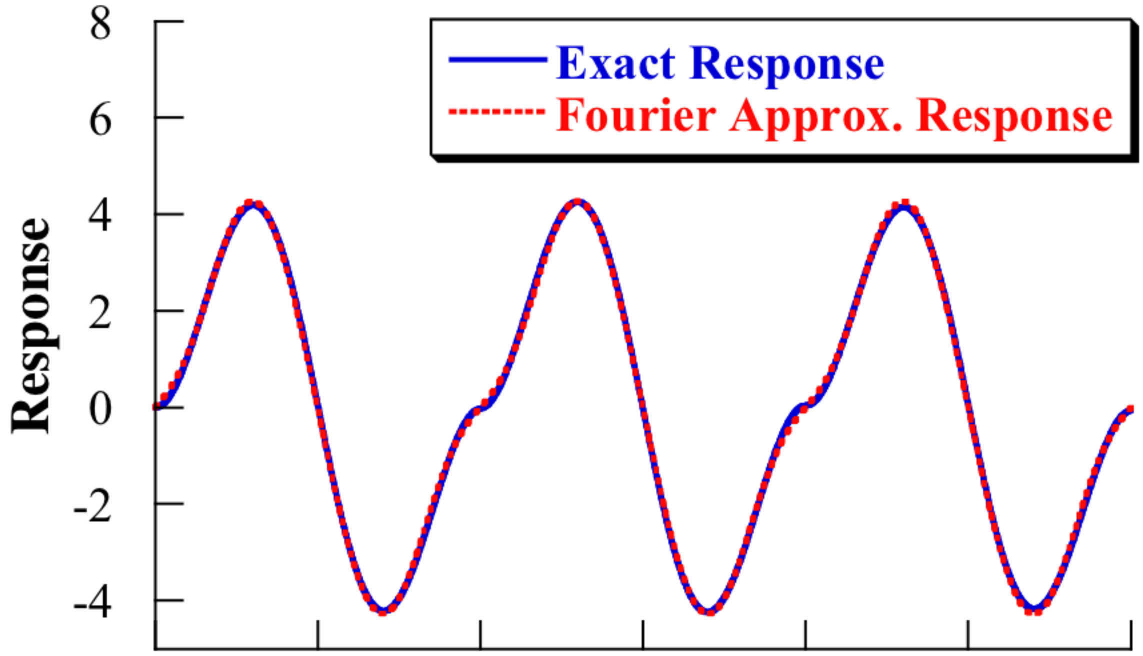


Problem 3.8 (cont.) 3-term Approximation



Even with just 3 terms, we get the main trends of the response.



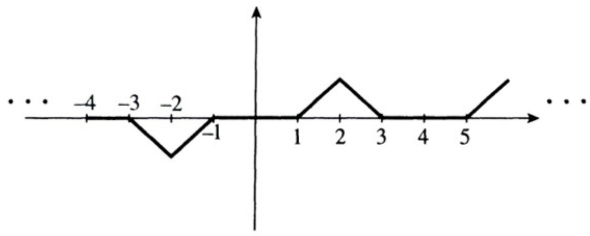


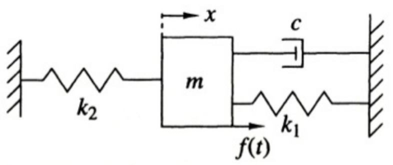
This is only 3 terms, but notice how close the response is still approximated.

Problem 3.9

3.9. Compute and plot the response for the system illustrated in Figure P3.9 for the input force also shown. You'll note that the second harmonic ($2\omega_0 = 1.57$) is very close to the system's natural frequency. Since the damping is equal to zero, why isn't this harmonic dominating the response?

Notice that this is the same input as Problem 3.3. We will use it and the transfer function of this system to get the response of this system.





$m = .4$ kg
$k_1 = .256$ N/m
$k_2 = .768$ N/m

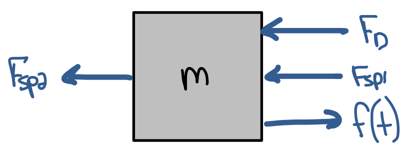
Figure P3.9

From Problem 3.3

$$f(t) = \sum_{n=1}^{\infty} \frac{4 f_{max}}{(n\pi)^2}\left[2\left(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right) + \left(-\sin\frac{n\pi}{4} - \frac{\sin 3n\pi}{4} + \sin\frac{5n\pi}{4} + \sin\frac{7n\pi}{4}\right)\right]\sin(n\omega_0 t)$$

Model the system to find the transfer function.

FBD



$$m\ddot{x} = -F_{sp1} - F_{sp2} - F_D + f(t) \longrightarrow m\ddot{x} = -(k_1+k_2)x - c\dot{x} + f(t)$$

$$m\ddot{x} + c\dot{x} + (k_1+k_2)x = f(t) \longrightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f}{m}$$

where $2\zeta\omega_n = \frac{c}{m}$ $\quad \omega_n^2 = \frac{k_1+k_2}{m}$

This is a standard direct-force mass-spring-damper system, so we know that the transfer function is:

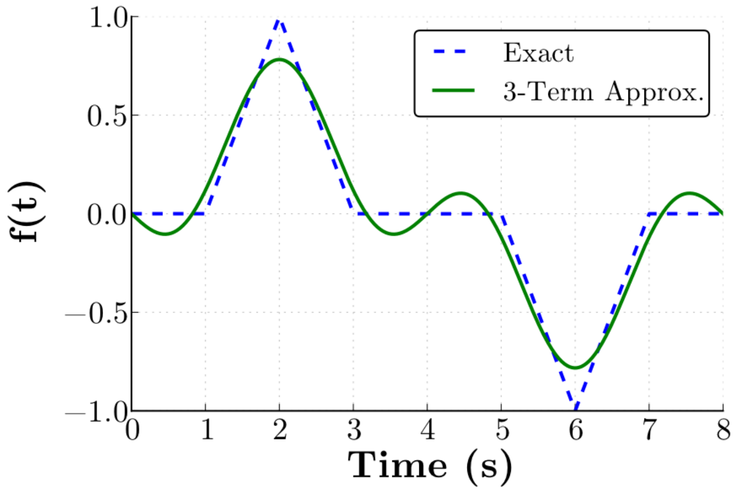
$$g(\omega) = \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}$$

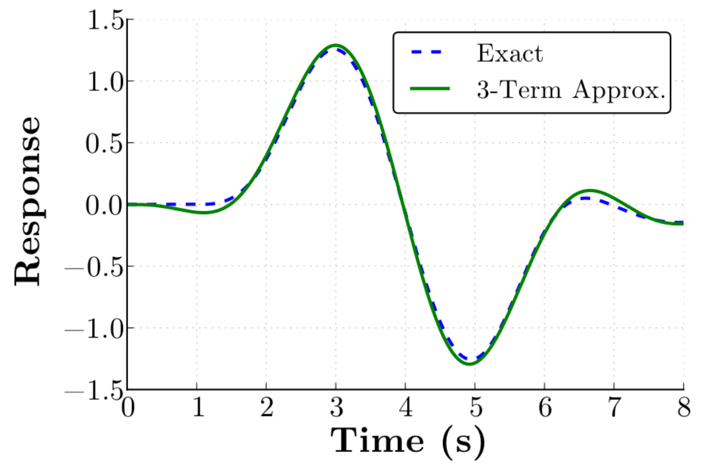
(If you don't recognize that, assume f(t) = $\bar{f}e^{i\omega t}$, x(t) = $\bar{x}e^{i\omega t}$ and solve for $\bar{x}/\bar{f}$.

So, the response is

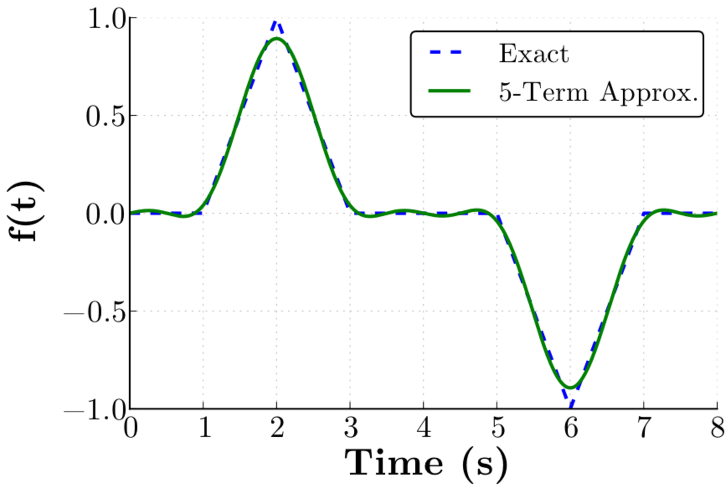
$$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{4 f_{max}}{(n\pi)^2}\left[2\left(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right) + \left(-\sin\frac{n\pi}{4} - \frac{\sin 3n\pi}{4} + \sin\frac{5n\pi}{4} + \sin\frac{7n\pi}{4}\right)\right]\left[\frac{1}{m\sqrt{(\omega_n^2-(n\omega_0)^2)^2-(2\zeta\omega_n n\omega_0)^2}}\right]\sin(n\omega_0 t)}$$

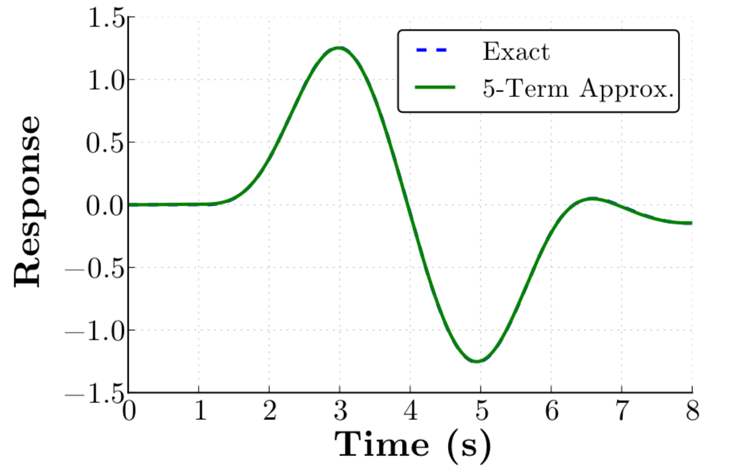
Problem 3.9 (cont).





We get pretty close even with this 3-term approximation.





With 5-terms, there is almost no difference between the exact and approximate solutions.

The questions also asks why the 2nd harmonic doesn't significantly contribute to the response. That is because b2 = 0.

Problem 3.15

3.15. Determine $x(2.)$ and $\dot{x}(2)$ for a direct force excited, spring-mass system for the input given in Figure P3.15 via a convolution approach. $x(0) = \dot{x}(0) = 0$, $m = 1$ kg, $k = 9$ N/m.

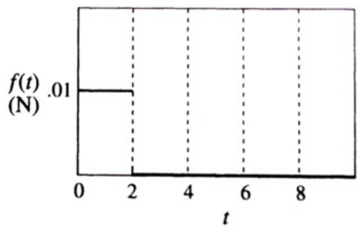
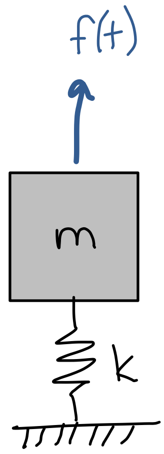


Figure P3.15



We know: $m\ddot{x} + kx = f(t)$

the impulse resp. for this system is $h(t) = \frac{1}{m\omega_n}\sin\omega_n t$ (see p.169 of book for deriv.)

$$f(t) = \begin{cases} \bar{f} & 0 \le t \le 2 \\ 0 & t > 2 \end{cases}$$

$$x(t) = \int_0^t h(\tau)\, f(t-\tau)\, d\tau$$

This is only valid when $t \le 2$

$$= \int_0^t \left(\frac{1}{m\omega_n}\sin\omega_n\tau\right)\bar{f}\, d\tau$$

$$= \frac{\bar{f}}{m\omega_n}\int_0^t \sin\omega_n\tau\, d\tau = \frac{-\bar{f}}{m\omega_n^2}\cos\omega_n\tau\Big|_0^t = \frac{-\bar{f}}{m\omega_n^2}\left(\cos\omega_n t - 1\right)$$

$$x(t) = \frac{\bar{f}}{m\omega_n^2}\left(1 - \cos\omega_n t\right)$$

$$\dot{x}(t) = \frac{\bar{f}}{m\omega_n}\sin\omega_n t$$

Remember that these are only valid $t \le 0.2s$

For this system $\bar{f} = .01\,\text{N}$ and $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{1}} = 3$

Filling in these values at $t = 2s$, we find:

$$\boxed{\begin{aligned} x(2) &= 4.43 \times 10^{-5}\ \text{m} \\ \dot{x}(2) &= -0.000931\ \text{m/s}^2 \end{aligned}}$$

If we wanted to write the solution for t>2s, we would use these as the initial conditions for the free response of the mass-spring system.

Problem 3.18

3.18. Will the force input illustrated in Figure P3.18 lead to a stable oscillation (no growth in amplitude) for a direct force excited, spring-mass system for which $m = 100$ kg and $k = 2500\pi^2$ N/m? Use a convolution approach.

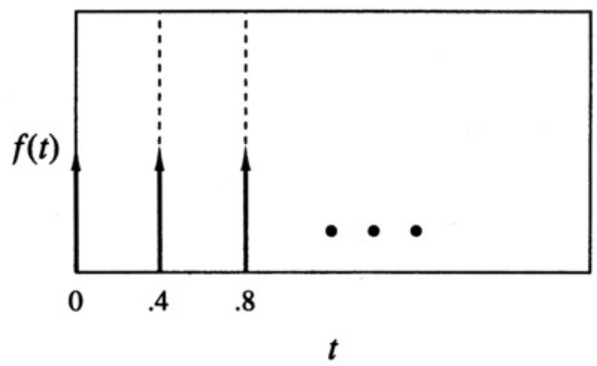
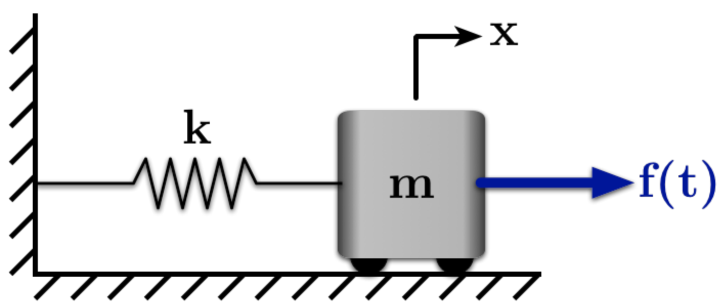


Figure P3.18

A direct-force spring-mass system:



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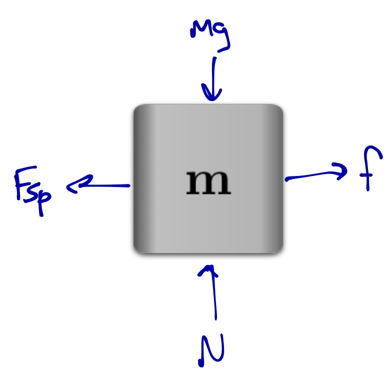

$$m\ddot{x} = -F_{sp} + f = -kx + f$$

$m\ddot{x} + kx = f \rightarrow \ddot{x} + \omega_n^2 x = \frac{f}{m}$ ← For the parameters given $\omega_n = \sqrt{\frac{2500\pi^2}{100}} = \sqrt{25\pi^2} = 5\pi \ \frac{rad}{s}$

or $f = \frac{5\pi}{2\pi}$ Hz $= 2.5$ Hz so $\tau_n = 0.4$ s

Since frequency of the input matches the natural freq of the system, the response will grow to ∞.

Let's prove this using the convolution integral, as the problem asks.

The impulse response without damping ($\zeta = 0$) is: $h(t) = \frac{1}{m\omega_n}\sin(\omega_n t)$

So, the convolution integral becomes:

$$x(t) = \int_0^t f(t-\tau)h(\tau)d\tau = \int_0^t f(t-\tau)\left(\frac{1}{m\omega_n}\sin\omega_n\tau\right)d\tau = \frac{1}{m\omega_n}\int_0^t f(t-\tau)\sin(\omega_n\tau)d\tau$$

Now, we just need to define what $f(t)$ is.

$$f(t) = \delta(t - n\tau_n) \quad , n = 0,1,2,\ldots$$ where $\tau_n \equiv$ period of oscillation of the system $\left(\tau_n = \frac{2\pi}{\omega_n}\right)$

$$= \delta\left(t - \frac{2n\pi}{\omega_n}\right)$$

Problem 3.18 (cont.)

$$x(t)=\frac{1}{m\omega_n}\int_0^t f(t-\tau)\sin(\omega_n\tau)\,d\tau \quad \text{where} \quad f(t)=\delta\left(t-\frac{2n\pi}{\omega_n}\right) \quad n=0,1,2,\ldots \Rightarrow f(t-\tau)=\delta\left(t-\tau-\frac{2n\pi}{\omega_n}\right) \quad n=0,1,\ldots$$

or

$$x(t)=\frac{1}{m\omega_n}\int_0^t f(\tau)\sin(\omega_n(t-\tau))\,d\tau \quad \text{where} \quad f(t)=\delta\left(t-\frac{2n\pi}{\omega_n}\right) \quad n=0,1,2,\ldots$$

To solve this, let's look at the response after the 3^{rd} impulse, so

For $\frac{6\pi}{\omega_n}\le t\le\frac{8\pi}{\omega_n}$ $(0.6s\le t<0.8s)$

We know the response is just the sum of the responses to the impulses, so

1^{st} impulse response $(n=0)$

$$x_1(t)=\frac{1}{m\omega_n}\int_0^t \delta(t-\tau)\sin(\omega_n\tau)\,d\tau = \frac{-1}{m\omega_n^2}\cos(\omega_n\tau)\Big|_0^t = \frac{1}{m\omega_n^2}-\frac{1}{m\omega_n^2}\cos(\omega_n t)$$

2^{nd} impulse response $(n=1)$

$$x_2(t)=\frac{1}{m\omega_n}\int_{2\pi/\omega_n}^{t} \delta(t-\tau)\sin(\omega_n t)\,d\tau = \frac{-1}{m\omega_n^2}\cos(\omega_n\tau)\Big|_{2\pi/\omega_n}^{t} = \underbrace{\frac{1}{m\omega_n^2}-\frac{1}{m\omega_n^2}\cos(\omega_n t)}_{\substack{\text{True for } t>\frac{2\pi}{\omega_n} \\ =0 \text{ otherwise}}}$$

3^{rd} impulse response $(n=2)$

$$x_3(t)=\frac{1}{m\omega_n}\int_{4\pi/\omega_n}^{t} \delta(t-\tau)\sin(\omega_n\tau)\,d\tau = \frac{-1}{m\omega_n^2}\cos(\omega_n\tau)\Big|_{4\pi/\omega_n}^{t} = \underbrace{\frac{1}{m\omega_n^2}-\frac{1}{m\omega_n^2}\cos(\omega_n t)}_{\substack{\text{True for } t>\frac{4\pi}{\omega_n} \\ =0 \text{ otherwise}}}$$

This trend continues, so the total response is

$$x(t)=x_1(t)+\underbrace{x_2\left(t-\frac{2\pi}{\omega_n}\right)}_{\substack{=x_2(t) \text{ if } t>\frac{2\pi}{\omega_n} \\ =0 \text{ otherwise}}}+\underbrace{x_3\left(t-\frac{4\pi}{\omega_n}\right)}_{\substack{=x_3(t) \text{ if } t>\frac{4\pi}{\omega_n} \\ =0 \text{ otherwise}}}+\ldots$$

So, the response grows with each impulse