MCHE 485: Mechanical Vibrations
Spring 2019 – Homework 2

Assigned: Thursday, February 7th
Due: Friday, February 15th, 5pm

Assignment: From "Principles of Vibration" by Benson Tongue, problems:
1.70, 1.88, 2.9, 2.24, 2.32, 2.34, 2.42, 2.51, 2.60

Submission: Emailed *single* pdf document:

- to `joshua.vaughan@louisiana.edu`
- with subject line and filename `ULID-MCHE485-HW2`, where `ULID` is your ULID
- *Note:* Submissions with incorrect filenames or submitted as multiple images/pdfs will be rejected.

Problem 1.70

1.70. Find the equilibrium position for the system illustrated in Figure P1.70 and compute the natural frequency of oscillations about this equilibrium position. $m_1 = 1$ kg, $l_1 = 1.2$ m, $l_2 = .1$ m, $k = 6000$ N/m.

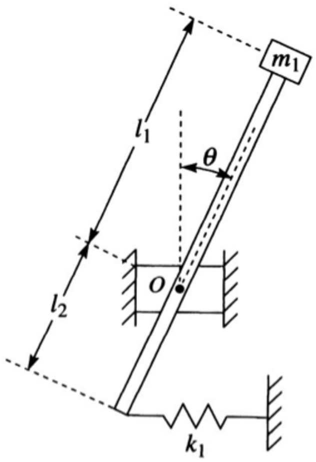

Figure P1.70

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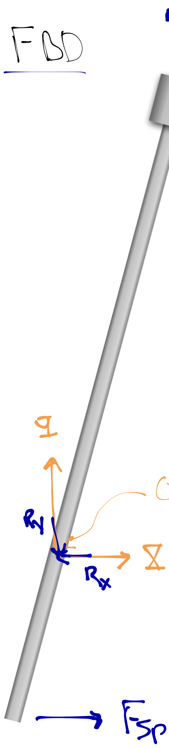


Sum moments about point O

$$I_o\bar{\alpha} = \sum \bar{M}_o$$

Assume small angles of θ such that F_{sp} is always in the $\bar{I}$ direction

$$F_{sp} = k_1 l_2 \sin\theta$$

$$\sum \bar{M}_o = \left(\bar{r}_{m_1/o} \times -m_1 g\bar{J}\right) + \left(\bar{r}_{sp/o} \times F_{sp}\bar{I}\right)$$

$$= \left[\left(l_1 \sin\theta \bar{I} + l_1 \cos\theta \bar{J}\right) \times \left(-m_1 g \bar{J}\right)\right] + \left[\left(-l_2 \sin\theta \bar{I} - l_2 \cos\theta \bar{J}\right) \times \left(k_1 l_2 \sin\theta\ \bar{I}\right)\right]$$

$$= -m_1 g\, l_1 \sin\theta\, \bar{K} + \left(-k_1 l_2^2 \cos\theta \sin\theta\right)\bar{K}$$ ← We could work straight from here. These moments balance at equil.

$$\bar{\alpha} = -\ddot{\theta}\bar{K}$$ (in the frame we've chosen a positive $\ddot{\theta}$ is in the $-\bar{K}$ direction)

$$I_o = m_1 l_1^2$$

$$-m_1 l_1^2 \ddot{\theta} = -m_1 g l_1 \sin\theta + k_1 l_2^2 \cos\theta \sin\theta$$

$$m_1 l_1^2 \ddot{\theta} + k_1 l_2^2 \cos\theta \sin\theta - m_1 g l_1 \sin\theta = 0$$ ← nonlinear equation of motion

Eliminate/ignore the motion terms and find the θ that satisfies the equation

$$\cancel{m_1 l_1^2 \ddot{\theta}} + k_1 l_2^2 \cos\theta \sin\theta - m_1 g l_1 \sin\theta = 0$$

$$\underbrace{\left(k_1 l_2^2 \cos\theta - m_1 g l_1\right)}\ \underbrace{\sin\theta} = 0$$ ← either term can be 0 to make the equation hold

$$k_1 l_2^2 \cos\theta - m_1 g l_1 = 0$$

$$\boxed{\theta_{eq2} = \cos^{-1}\left(\frac{m_1 g l_1}{k_1 l_2^2}\right)}$$

$$\sin\theta = 0 \longrightarrow \boxed{\theta_{eq1} = N\pi \text{ where } N \text{ is an integer}}$$

Problem 1.88

1.88. Determine the period of oscillation for the system illustrated in Figure P1.88. (Consider small oscillations, about $\theta = 0$.) $m_1 = 1$ kg, $m_2 = 5$ kg, $l_1 = 2.4$ m, $l_2 = 4$ m.

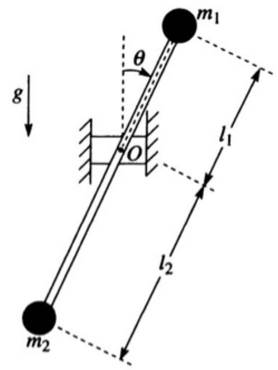
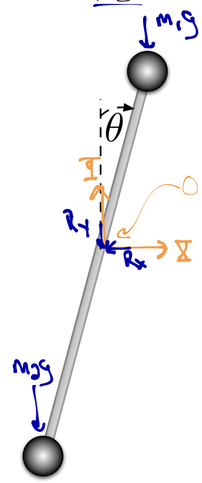


Figure P1.88

FBD



Sum moments about point O

$$I_O \bar{\alpha} = \sum M_O \qquad \bar{\alpha} = -\ddot{\theta} K \quad I_O = m_1 l_1^2 + m_2 l_2^2$$

$$\sum M_O = \left(\bar{r}_{m_1/O} \times -m_1 g \bar{J}\right) + \left(\bar{r}_{m_2/O} \times -m_2 g \bar{J}\right)$$

$$= \left[\left(l_1 \sin\theta \bar{I} + l_2 \cos\theta \bar{J}\right) \times \left(-m_1 g \bar{J}\right)\right] + \left[\left(-l_2 \sin\theta \bar{I} - l_2 \cos\theta \bar{J}\right) \times \left(-m_2 g \bar{J}\right)\right]$$

$$= \left(-m_1 g l_1 \sin\theta \bar{K}\right) + \left(m_2 g l_2 \sin\theta K\right)$$

$$\left[-I_O \ddot{\theta} = -m_1 g l_1 \sin\theta + m_2 g l_2 \sin\theta\right] \bar{K}$$

$$I_O \ddot{\theta} + \left(m_2 g l_2 - m_1 g l_1\right) \sin\theta = 0 \longleftarrow$$ Assume small angles about $\theta = 0$, so $\sin\theta \approx \theta$

$$I_O \ddot{\theta} + \left(m_2 g l_2 - m_1 g l_1\right) \theta = 0$$

$$\ddot{\theta} + \underbrace{\left(\frac{m_2 g l_2 - m_1 g l_1}{m_1 l_1^2 + m_2 l_2^2}\right)}_{= \omega_n^2} \theta = 0$$

$$\longrightarrow \boxed{\omega_n = \sqrt{\frac{m_2 g l_2 - m_1 g l_1}{m_1 l_1^2 + m_2 l_2^2}}}$$

Problem 2.9

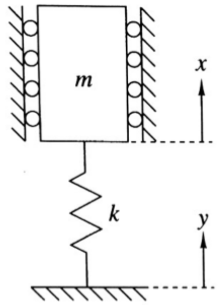


Figure P2.8

2.9. The spring-mass system shown in Figure P2.8 experiences a 2 mm in-phase oscillation when the base is oscillating at 300 rad/s with an amplitude of 1.1 mm. What is the static deflection of the mass?

equilibrium where spring force and gravity balance

This problem gives us a relationship between the input and output amplitude and phase. The transfer function expresses that as well, so we'll use it to determine the parameters we need.

We also know that at static displacement, the system is at equilibrium. The spring force and gravity are balancing.

At equil. $mg = kx \rightarrow$ so $x = \frac{m}{k} g$ ← $\frac{1}{\omega_n^2}$ because $\omega_n^2 = \frac{k}{m}$ ← So, we need to use the frequency response info to find ω_n^2

We know that the transfer function for this system is: $G(\omega) = \frac{\bar{X}}{\bar{Y}} = \frac{\omega_n^2}{\omega_n^2 - \omega^2}$

The problem tells us that $|\bar{X}| = 2mm$, and in phase tells us $\bar{X} = +2mm$

$|\bar{Y}| = 1.1mm$ and $\omega = 300 \frac{rad}{s}$

$$G(\omega) = \frac{2}{1.1} = \frac{\omega_n^2}{\omega_n^2 - (300)^2} \longrightarrow 1.82(\omega_n^2 - 90000) = \omega_n^2$$

$$0.82\,\omega_n^2 = 163636.36 \longrightarrow \omega_n^2 = 200000$$

So, the static deflection of the mass is:

$$x = \frac{1}{\omega_n^2} g = 0.00005 m$$

$$\boxed{x_{eq} = 0.05 mm}$$

Problem 2.24

2.24. Given the displacement input $y(t) = \bar{y}\cos(\omega t)$, determine the equation of motion for the system illustrated in Figure P2.24 (neglect gravity). What is the natural frequency for the system?

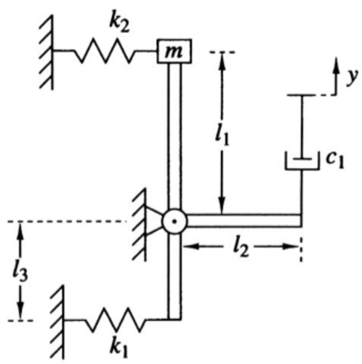
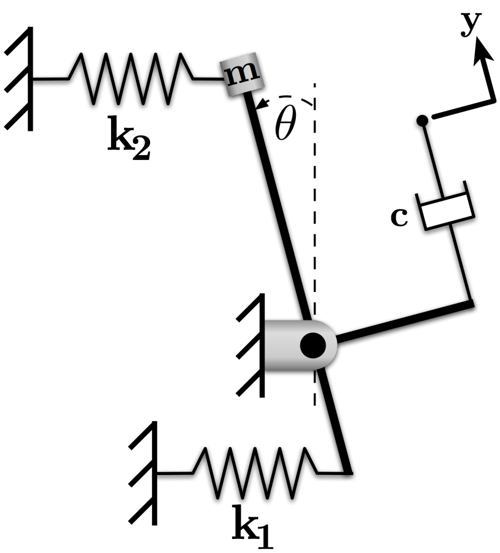
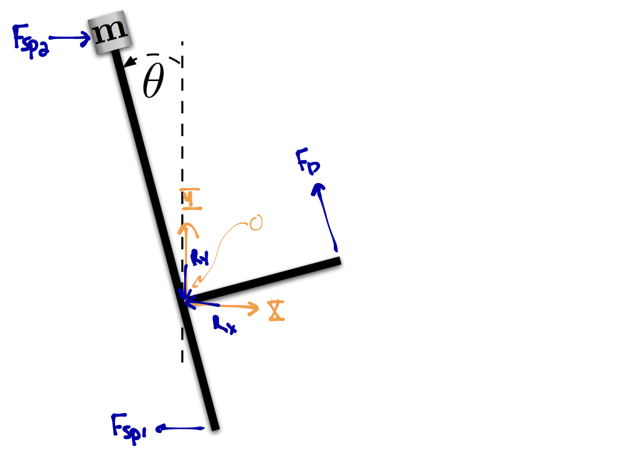


Figure P2.24





Assume small θ, so
$$F_{sp1} = k_1 l_3 \sin\theta$$
$$F_{sp2} = k_2 l_1 \sin\theta$$
$$F_D = c\left(\dot{y} - \underbrace{l_3\dot{\theta}\cos\theta}_{\text{velocity of the damper attachment point}}\right)$$

Sum moments about point O

$$I_O\bar{\alpha} = \Sigma\bar{M}_O \qquad I_O = m l_1^2 \qquad \alpha = \ddot{\theta}\bar{K}$$

$$\Sigma\bar{M}_O = \left(\bar{r}_{sp1/O} \times -F_{sp1}\bar{I}\right) + \left(\bar{r}_{D/O} \times F_D\bar{J}\right) + \left(\bar{r}_{sp2/O} \times F_{sp2}\bar{I}\right)$$

$$= \left[\left(-l_3\sin\theta\bar{I} - l_3\cos\theta\bar{J}\right) \times \left(-k_1 l_3 \sin\theta\bar{I}\right)\right] + \left[\left(l_2\cos\theta\bar{I} + l_2\sin\theta\bar{J}\right) \times c\left(\dot{y} - l_2\dot{\theta}\cos\theta\right)\bar{J}\right]$$
$$+ \left[\left(-l_1\sin\theta\bar{I} + l_1\cos\theta\bar{J}\right) \times \left(k_2 l_1 \sin\theta\bar{I}\right)\right]$$

$$= \left[-k_1 l_3^2\cos\theta\sin\theta\bar{K}\right] + \left[c l_2\cos\theta\left(\dot{y} - l_2\dot{\theta}\cos\theta\right)\bar{K}\right] + \left[-k_2 l_1^2\cos\theta\sin\theta\bar{K}\right]$$

$$\boxed{m l_1^2\ddot{\theta} = -\left(k_1 l_3^2 + k_2 l_1^2\right)\cos\theta\sin\theta - c l_2^2\dot{\theta}\cos^2\theta + c l_2\dot{y}\cos\theta} \leftarrow \text{Nonlinear Equation of motion}$$

Problem 2.24 (cont.)

$$m l_1^2 \ddot{\theta} = -\left(k_1 l_3^2 + k_2 l_1^2\right)\cos\theta\sin\theta - c l_2^2 \dot{\theta}\cos^2\theta + c l_2 \dot{y}\cos\theta$$

$\leftarrow$ linearize by assuming small angles
$\sin\theta \approx \theta$ and $\cos\theta = 1$ ($\cos^2\theta = 1$ too)

$$m l_1^2 \ddot{\theta} = -\left(k_1 l_3^2 + k_2 l_1^2\right)\theta - c l_2^2 \dot{\theta} + c l_2 \dot{y}$$

$$\boxed{m_1 l_1^2 \ddot{\theta} + c l_2^2 \dot{\theta} + \left(k_1 l_3^2 + k_2 l_1^2\right)\theta = c l_2 \dot{y}}$$

$\leftarrow$ Linearized equation of motion

Divide by $m_1 l_1^2$

$$\ddot{\theta} + \frac{c l_2^2}{m l_1^2}\dot{\theta} + \underbrace{\left(\frac{k_1 l_3^2 + k_2 l_1^2}{m_1 l_1^2}\right)}_{=\omega_n^2}\theta = \frac{c l_2}{m l_1^2}\dot{y}$$

$$\boxed{\omega_n^2 = \sqrt{\frac{k_1 l_3^2 + k_2 l_1^2}{m_1 l_1^2}}}$$

Problem 2.32

2.32. Derive the equations of motion for the system shown in Figure P2.32 for a displacement input given by $y(t) = \bar{y}\cos(\omega t)$ (neglect gravity).

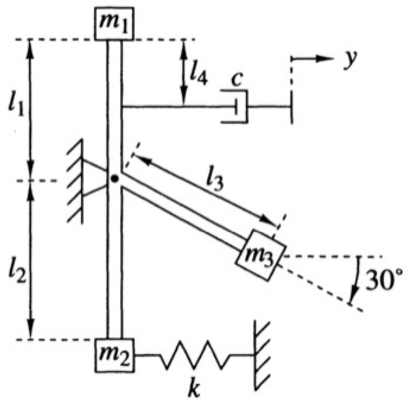
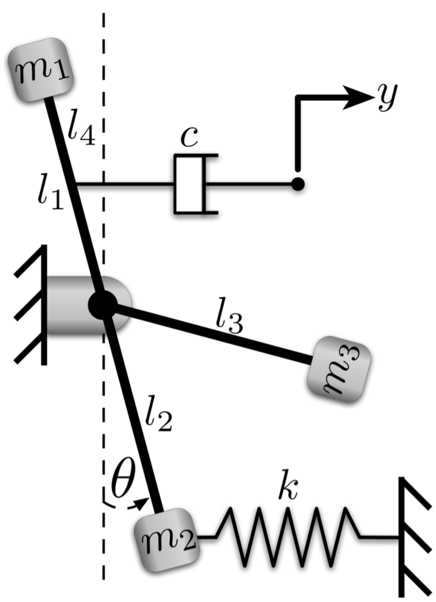
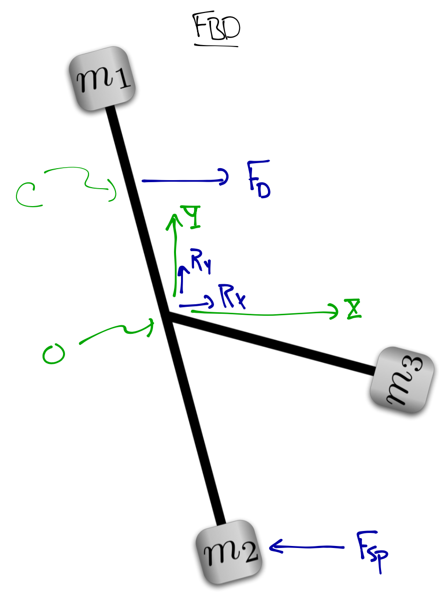


Figure P2.32





Use Θ to describe the system motion

We'll assume small angles of Θ about $\Theta = 0$. This means that both the spring and damper forces will act only in the X-direction.

$$F_{sp} = k\delta_1 \quad \rightarrow \quad \delta_1 = l_2 \sin\Theta \quad \rightarrow \quad F_{sp} = k l_2 \sin\Theta$$

$$F_D = c\dot{\delta}_2 \qquad \delta_2 = y + (l_1 - l_4)\sin\Theta$$

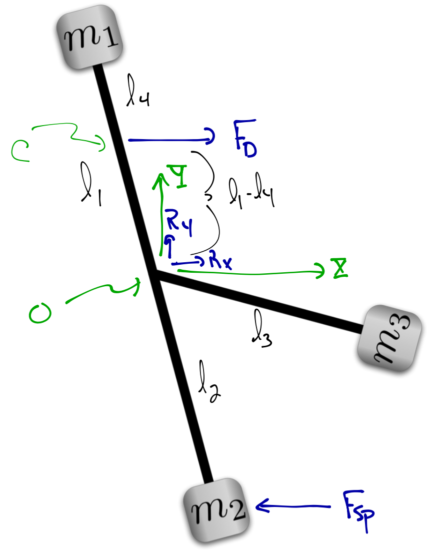
This is the displacement of the damper connection to the rod

+ because a positive Θ increases δ_2

$$\text{So} \quad \dot{\delta}_2 = \dot{y} + (l_1 - l_4)\dot{\Theta}\cos\Theta$$

$$F_D = c\left(\dot{y} + (l_1 - l_4)\dot{\Theta}\cos\Theta\right)$$

Problem 2.32 (cont.)



$$F_{sp} = k l_2 \sin\theta$$

$$F_D = c\left(\dot{y} + (l_1 - l_4)\dot{\theta}\cos\theta\right)$$

$$I_O = m_1 l_1^2 + m_2 l_2^2 + m_3 l_3^2$$ ← moment of inertia about O
It's just 3 point masses in pure rotation

Using Newton/Euler

Sum moments about point O. The system is in pure rotation about that point

$$\sum \bar{M}_O = \left(\bar{r}_{m_2/O} \times \bar{F}_{sp}\right) + \left(\bar{r}_{c/O} \times \bar{F}_D\right)$$

$$= \left[\left(l_2 \sin\theta \bar{I} - l_2 \cos\theta \bar{J}\right) \times \left(-k l_2 \sin\theta \bar{I}\right)\right] + \left[\left(-(l_1 - l_4)\sin\theta \bar{I} + (l_1 - l_4)\cos\theta \bar{J}\right) \times \left(c\left(\dot{y} + (l_1 - l_4)\dot{\theta}\cos\theta \bar{I}\right)\right)\right]$$

$$= \left[-k l_2^2 \sin\theta\cos\theta \, \underline{K}\right] + \left[-c(l_1 - l_4)\cos\theta\left(\dot{y} + (l_1 - l_4)\dot{\theta}\cos\theta\right)\underline{K}\right]$$

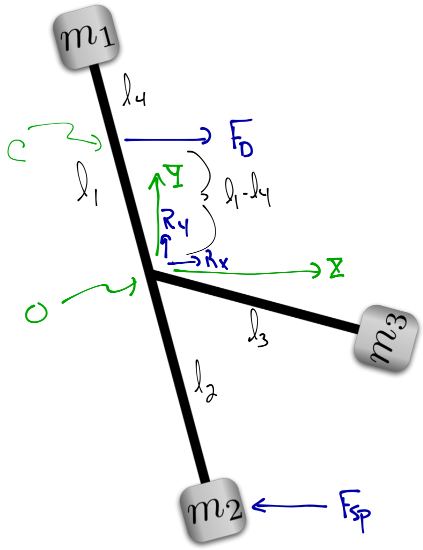
$$\sum \bar{M}_O = \dot{\bar{H}}_O \Rightarrow \sum \bar{M}_O = I_O \ddot{\theta}\, \underline{K}$$

All in $\underline{K}$, as they should be

$$I_O \ddot{\theta} = -k l_2^2 \sin\theta\cos\theta - c\left((l_1 - l_4)\dot{y}\cos\theta + (l_1 - l_4)^2 \dot{\theta}\cos^2\theta\right)$$ ← In $\underline{K}$-direction

$$\boxed{I_O \ddot{\theta} + c(l_1 - l_4)^2 \dot{\theta}\cos^2\theta + k l_2^2 \sin\theta\cos\theta = -c(l_1 - l_4)\dot{y}\cos\theta}$$

Problem 2.32 (cont.)



$$F_{sp} = k l_2 \sin\theta$$

$$F_D = c\left(\dot{y} + (l_1 - l_4)\dot{\theta}\cos\theta\right)$$

$$I_O = m_1 l_1^2 + m_2 l_2^2 + m_3 l_3^2$$ ← moment of inertia about O
It's just 3 point masses in pure rotation

Using Lagrange's Method

The system is in pure rotation about point O, so we can use the rotational form of Kinetic Energy

$$T = \tfrac{1}{2} I_O \dot{\theta}^2$$

$$U = U_{sp} = \tfrac{1}{2} k \delta_1^2 = \tfrac{1}{2} k (l_2 \sin\theta)^2$$

$$RD = \tfrac{1}{2} c \dot{\delta}_2^2 = \tfrac{1}{2} c\left(\dot{y} + (l_1 - l_4)\dot{\theta}\cos\theta\right)^2$$

$$L = T - U = \tfrac{1}{2} I_O \dot{\theta}^2 - \tfrac{1}{2} k (l_2 \sin\theta)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \frac{\partial RD}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_1$$

$Q_1 = 0$ ← we have no non-conservative external forces

$$\frac{\partial L}{\partial \dot{\theta}} = I_O \dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I_O \ddot{\theta}$$

$$\frac{\partial RD}{\partial \dot{\theta}} = c\left((l_1 - l_4)\cos\theta\right)\left(\dot{y} + (l_1 - l_4)\dot{\theta}\cos\theta\right) = c(l_1 - l_4)\dot{y}\cos\theta + c(l_1 - l_4)^2\dot{\theta}\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -k(l_2\cos\theta)(l_2\sin\theta) = -k l_2^2 \sin\theta\cos\theta$$

$$I_O\ddot{\theta} + c(l_1 - l_4)\dot{y}\cos\theta + c(l_1 - l_4)^2\dot{\theta}\cos\theta + k l_2^2 \sin\theta\cos\theta = 0$$

$$\boxed{I_O\ddot{\theta} + c(l_1 - l_4)^2\dot{\theta}\cos^2\theta + k l_2^2 \sin\theta\cos\theta = -c(l_1 - l_4)\dot{y}\cos\theta}$$ ← Same as Newton/Euler as it better be!

Problem 2.34

2.34. In Figure P2.34, does the force transmitted to the floor reach a maximum as ω is increased or does it increase indefinitely? At high ω what is more important to the transmitted force, k or c? ($x = x_0 \sin(\omega t)$.)

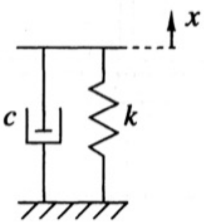
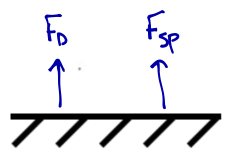


Figure P2.34

FBD



Forces on the floor are just from the spring and damper

$$F_{floor} = F_D + F_{sp} \qquad F_D = c\dot{x} \qquad F_{sp} = kx$$

$$= c\dot{x} + kx$$

We're given that $x(t) = x_0 \sin(\omega t)$ so $\dot{x}(t) = \omega x_0 \cos(\omega t)$

So,

$$F_{floor} = \underbrace{c\left(\omega x_0 \cos(\omega t)\right)} + k\left(x_0 \sin(\omega t)\right)$$

As $\omega \to \infty$ this term dominates, so damping force is more important

We could also collect terms by remembering

$$a\sin(\omega t) + b\cos(\omega t) = \sqrt{a^2+b^2}\,\sin(\omega t - \phi) \quad \text{where } \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

So

$$F_{sp} = \left[x_0\sqrt{(c\omega)^2 + k^2}\,\right] \sin(\omega t + \phi) \quad \text{where } \phi = \tan^{-1}\left(\frac{c\omega}{k}\right)$$

we see again that the damping term dominates as $\omega \to \infty$

Problem 2.42

2.42. Consider the pendular system illustrated in Figure P2.42 (lumped mass m on the end of a rigid rod of length l). A small motor at O produces a sinusoidally varying torque ($M = \bar{M}\sin(\omega t)$) that acts on the pendulum. Find the transfer function from input torque to response angle $\theta(t)$. Linearize your system equations about $\theta = 0$.

We set this one up in one of the first lectures.

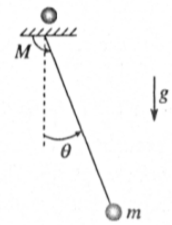
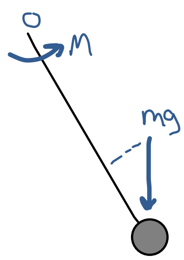


Figure P2.42

FBD



Sum moments about point O

$$I\ddot{\theta} = \sum \bar{M}_o = -mgl\sin\theta + M$$

$$ml^2\ddot{\theta} + mgl\sin\theta = M$$

assume small angles

$$ml^2\ddot{\theta} + mgl\theta = M$$

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{M}{ml^2}$$

Assume $\theta(t) = \bar{\theta}\sin(\omega t)$ ← match the form of the input

$$\dot{\theta}(t) = \omega\bar{\theta}\cos(\omega t)$$

$$\ddot{\theta}(t) = -\omega^2\bar{\theta}\sin(\omega t)$$

plug these back into the Eq. of Motion

$$-\omega^2\bar{\theta}\sin\omega t + \underbrace{\frac{g}{l}}_{\omega_n^2}\bar{\theta}\sin\omega t = \frac{\bar{M}}{ml^2}\sin\omega t$$

$$(-\omega^2 + \omega_n^2)\bar{\theta}\cancel{\sin\omega t} = \frac{\bar{M}}{ml^2}\cancel{\sin\omega t}$$

$$\boxed{\frac{\bar{\theta}}{\bar{M}} = \frac{1/ml^2}{\omega_n^2 - \omega^2}}$$

Notice that this has the same form as the direct force linear response.

You could have recognized that and jumped directly here.

Problem 2.51

2.51. Find the complex transfer function between the velocity input $\dot{y}$ and the displacement output x for the system illustrated in Figure P2.51.

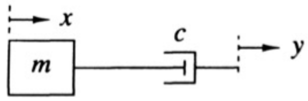
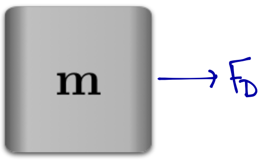


Figure P2.51

FBD



$$m\ddot{x} = F_D = c(\dot{y} - \dot{x})$$

$$m\ddot{x} + c\dot{x} = c\dot{y} \longrightarrow \ddot{x} + \frac{c}{m}\dot{x} = \frac{c}{m}\dot{y}$$

Assume $y(t) = \bar{Y}e^{i\omega t}$ and $x(t) = \bar{X}e^{i\omega t}$

Plug into the equations of motion

$$\left(-\omega^2 + \frac{c}{M}i\omega\right)\bar{X}\cancel{e^{i\omega t}} = \frac{c}{m}i\omega\bar{Y}\cancel{e^{i\omega t}}$$

$$\boxed{\frac{\bar{X}}{\bar{Y}} = \frac{i\frac{c}{M}}{-\omega^2 + i\frac{c}{m}\omega}}$$

Problem 2.60

2.60. Assume that a seismically excited system is accurately modeled by as shown in Figure P2.1. You need to determine whether the weld connecting the spring to the mass will hold. The maximum stress that the weld can withstand is $1.0 \times 10^6\ \mathrm{N/m^2}$. The mass is 10 kg and the spring constant is 4000 N/m. The damping constant is 40 N·s/m. Is it reasonable to allow the excitation frequency to reach 10 rad/s? The attachment area of the spring is 1 mm × 1 mm and the base amplitude is 1 mm.

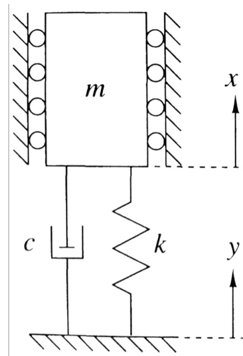
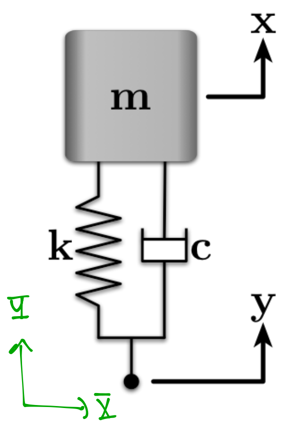
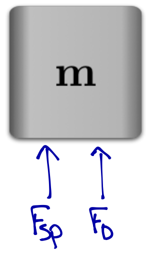


Figure P2.1



FBD



$F_{sp} = k(y-x)$

$F_D = c(\dot{y}-\dot{x})$

$\sum \bar{F} = m\bar{a}$

$k(y-x) + c(\dot{y}-\dot{x}) = m\ddot{x}$ ← all in $\bar{J}$ direction

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y$$

Equation of motion

$2\zeta\omega_n = \frac{c}{m}$ and $\omega_n^2 = \frac{k}{m}$ ← $2\zeta\omega_n = \frac{40}{10} = 4$ $\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20\ \mathrm{rad/s}$

Assuming a pure harmonic input, we know $|x(t)|$ is

$$|x(t)| = \frac{\sqrt{\omega_n^4 + (2\zeta\omega\omega_n)^2}}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2}}|Y|$$

This is from the transfer function for a seismically excited system

The spring force on the mass is then:

$$|F_{sp}| = k|x| = k(1.33|y|) \approx 5333.13|y| = 5.33\ \mathrm{N}$$

This force acts over an area that is

$$(1\mathrm{mm} \times 1\mathrm{mm}) = 1e^{-6}\ \mathrm{m} \longrightarrow$$

$$\boxed{\frac{|F_{sp}|}{1e^{-6}} \hat{=} 5.33e^{6}\ \mathrm{N/m^2}}$$

So, no the weld will not hold