3.38 Determine the Fourier series representation for the excitation of Fig. 3-31.

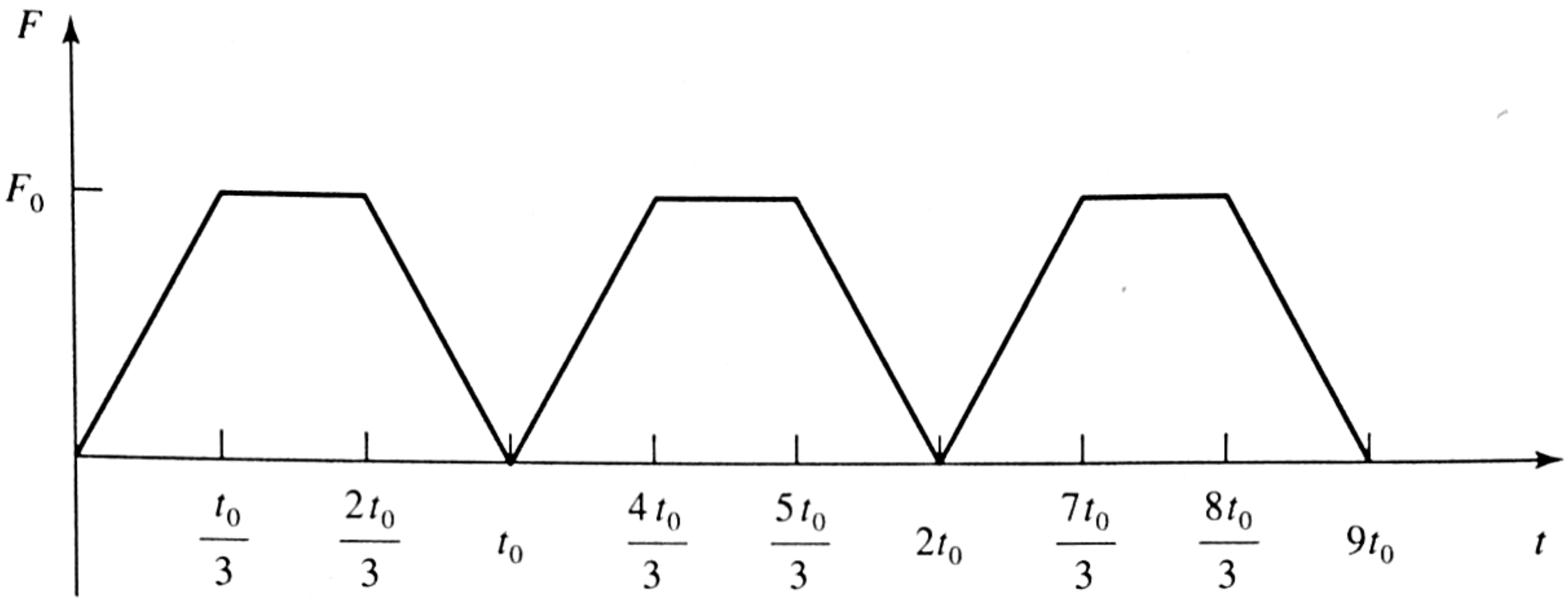

Fig. 3-31

The excitation of Fig. 3-31 is an even excitation of period t_0. Hence $b_i = 0$, $i = 1, 2, \ldots$. The Fourier cosine coefficients are

$$a_0 = \frac{2}{t_0}\int_0^{t_0} F(t)\,dt$$

$$= \frac{2}{t_0}\left[\int_0^{t_0/3}\left(\frac{3F_0}{t_0}\right)t\,dt + \int_{t_0/3}^{(2/3)t_0} F_0\,dt + \int_{(2/3)t_0}^{t_0} 3F_0\left(1-\frac{t}{t_0}\right)dt\right]$$

$$= \frac{4}{3}F_0$$

$$a_i = \frac{2}{t_0}\int_0^{t_0} F(t)\cos\frac{2\pi i}{t_0}t\,dt$$

$$= \frac{2}{t_0}\left[\int_0^{(1/3)t_0}\left(\frac{3F_0}{t_0}\right)t\cos\frac{2\pi i}{t_0}t\,dt + \int_{(1/3)t_0}^{(2/3)t_0} F_0\cos\frac{2\pi i}{t_0}t\,dt + \int_{(2/3)t_0}^{t_0} 3F_0\left(1-\frac{t}{t_0}\right)\cos\frac{2\pi i}{t_0}t\,dt\right]$$

$$= \frac{3F_0}{i^2\pi^2}\left(\frac{1}{2}\cos\frac{2\pi i}{3} + \frac{1}{2}\cos\frac{4\pi i}{3} - 1\right)$$

$$= \begin{cases} -\dfrac{9F_0}{2i^2\pi^2} & i = 1, 2, 4, 5, 7, 8, \ldots \\ 0 & i = 3, 6, 9, 12, \ldots \end{cases}$$

Thus the Fourier Series representation for $F(t)$ is

$$F(t) = \frac{2}{3}F_0 - \frac{9F_0}{2\pi^2}\sum_{i=1,2,4,5,7,8}^{\infty}\frac{1}{i^2}\cos\frac{2\pi i}{t_0}t$$

$$a_n = \frac{2}{N} \sum_{i=1}^{N} x_i \cos \frac{2n\pi t_i}{\tau} \tag{1.98}$$

$$b_n = \frac{2}{N} \sum_{i=1}^{N} x_i \sin \frac{2n\pi t_i}{\tau} \tag{1.99}$$

EXAMPLE 1.12 Fourier Series Expansion

Determine the Fourier series expansion of the motion of the valve in the cam-follower system shown in Fig. 1.53.

Solution: If $y(t)$ denotes the vertical motion of the pushrod, the motion of the valve, $x(t)$, can be determined from the relation:

$$\tan \theta = \frac{y(t)}{l_1} = \frac{x(t)}{l_2}$$

or

$$x(t) = \frac{l_2}{l_1} y(t) \tag{E.1}$$

where

$$y(t) = Y\frac{t}{\tau};\ 0 \leq t \leq \tau \tag{E.2}$$

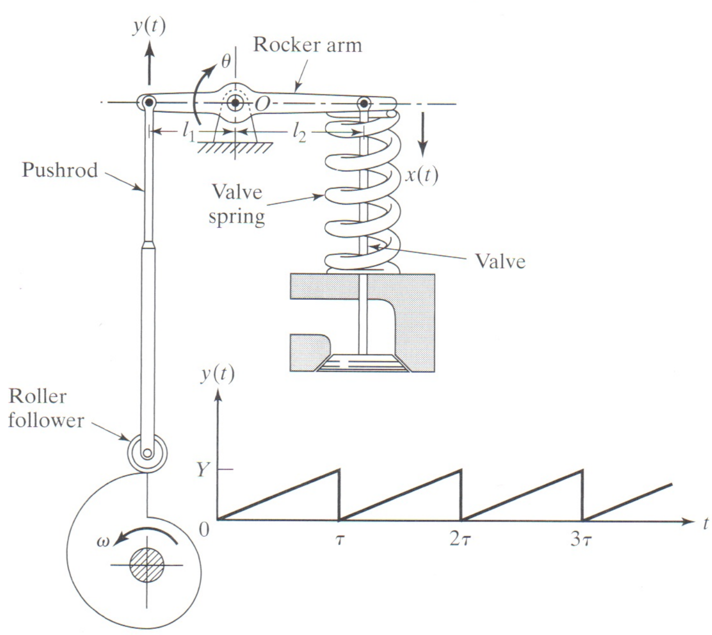

FIGURE 1.53 Cam-follower system.

and the period is given by $\tau = \dfrac{2\pi}{\omega}$. By defining

$$A = \frac{Yl_2}{l_1}$$

$x(t)$ can be expressed as

$$x(t) = A\frac{t}{\tau};\ 0 \le t \le \tau \tag{E.3}$$

Equation (E.3) is shown in Fig. 1.46(a). To compute the Fourier coefficients a_n and b_n, we use Eqs. (1.71) to (1.73):

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} A\frac{t}{\tau}\,dt = \frac{\omega}{\pi}\frac{A}{\tau}\left(\frac{t^2}{2}\right)_0^{2\pi/\omega} = A \tag{E.4}$$

$$\begin{aligned}
a_n &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos n\omega t\cdot dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} A\frac{t}{\tau}\cos n\omega t\cdot dt \\
&= \frac{A\omega}{\pi\tau}\int_0^{2\pi/\omega} t\cos n\omega t\cdot dt = \frac{A}{2\pi^2}\left[\frac{\cos n\omega t}{n^2} + \frac{\omega t\sin n\omega t}{n}\right]_0^{2\pi/\omega} \\
&= 0,\ n = 1, 2, \ldots
\end{aligned} \tag{E.5}$$

$$\begin{aligned}
b_n &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\sin n\omega t\cdot dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} A\frac{t}{\tau}\sin n\omega t\cdot dt \\
&= \frac{A\omega}{\pi\tau}\int_0^{2\pi/\omega} t\sin n\omega t\cdot dt = \frac{A}{2\pi^2}\left[\frac{\sin n\omega t}{n^2} - \frac{\omega t\cos n\omega t}{n}\right]_0^{2\pi/\omega} \\
&= -\frac{A}{n\pi},\ n = 1, 2, \ldots
\end{aligned} \tag{E.6}$$

Therefore the Fourier series expansion of $x(t)$ is

$$\begin{aligned}
x(t) &= \frac{A}{2} - \frac{A}{\pi}\sin\omega t - \frac{A}{2\pi}\sin 2\,\omega t - \ldots \\
&= \frac{A}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \ldots\right\}\right]
\end{aligned} \tag{E.7}$$

The first three terms of the series are shown plotted in Fig. 1.46(b). It can be seen that the approximation reaches the sawtooth shape even with a small number of terms.

■

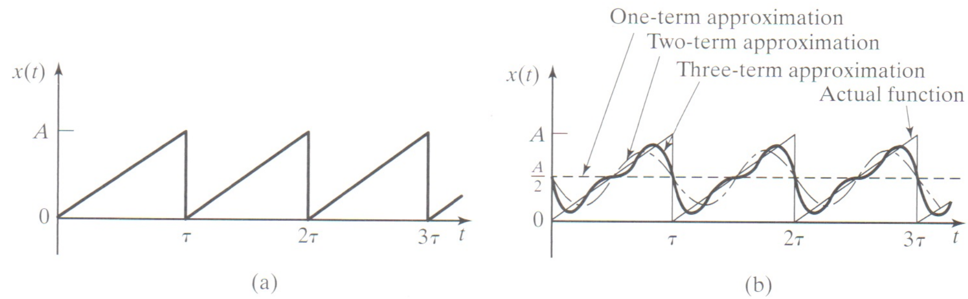

FIGURE 1.46 A periodic function.

1.11.1 Fourier Series Expansion

If $x(t)$ is a periodic function with period τ, its Fourier series representation is given by

$$\begin{aligned} x(t) &= \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\,\omega t + \cdots \\ &\quad + b_1 \sin \omega t + b_2 \sin 2\,\omega t + \cdots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \end{aligned} \tag{1.70}$$

where $\omega = 2\pi/\tau$ is the fundamental frequency and $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are constant coefficients. To determine the coefficients a_n and b_n, we multiply Eq. (1.70) by $\cos n\omega t$ and $\sin n\omega t$, respectively, and integrate over one period $\tau = 2\pi/\omega$; for example, from 0 to $2\pi/\omega$. Then we notice that all terms except one on the right-hand side of the equation will be zero, and we obtain

$$a_0 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t)\, dt = \frac{2}{\tau} \int_0^{\tau} x(t)\, dt \tag{1.71}$$

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \cos n\omega t\, dt = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t\, dt \tag{1.72}$$

$$b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \sin n\omega t\, dt = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t\, dt \tag{1.73}$$

The physical interpretation of Eq. (1.70) is that any periodic function can be represented as a sum of harmonic functions. Although the series in Eq. (1.70) is an infinite sum, we can approximate most periodic functions with the help of only a few harmonic functions. For example, the triangular wave of Fig. 1.46(a) can be represented closely by adding only three harmonic functions, as shown in Fig. 1.46(b).

Fourier series can also be represented by the sum of sine terms only or cosine terms only. For example, the series using cosine terms only can be expressed as

$$x(t) = d_0 + d_1 \cos(\omega t - \phi_1) + d_2 \cos(2\omega t - \phi_2) + \cdots \tag{1.74}$$