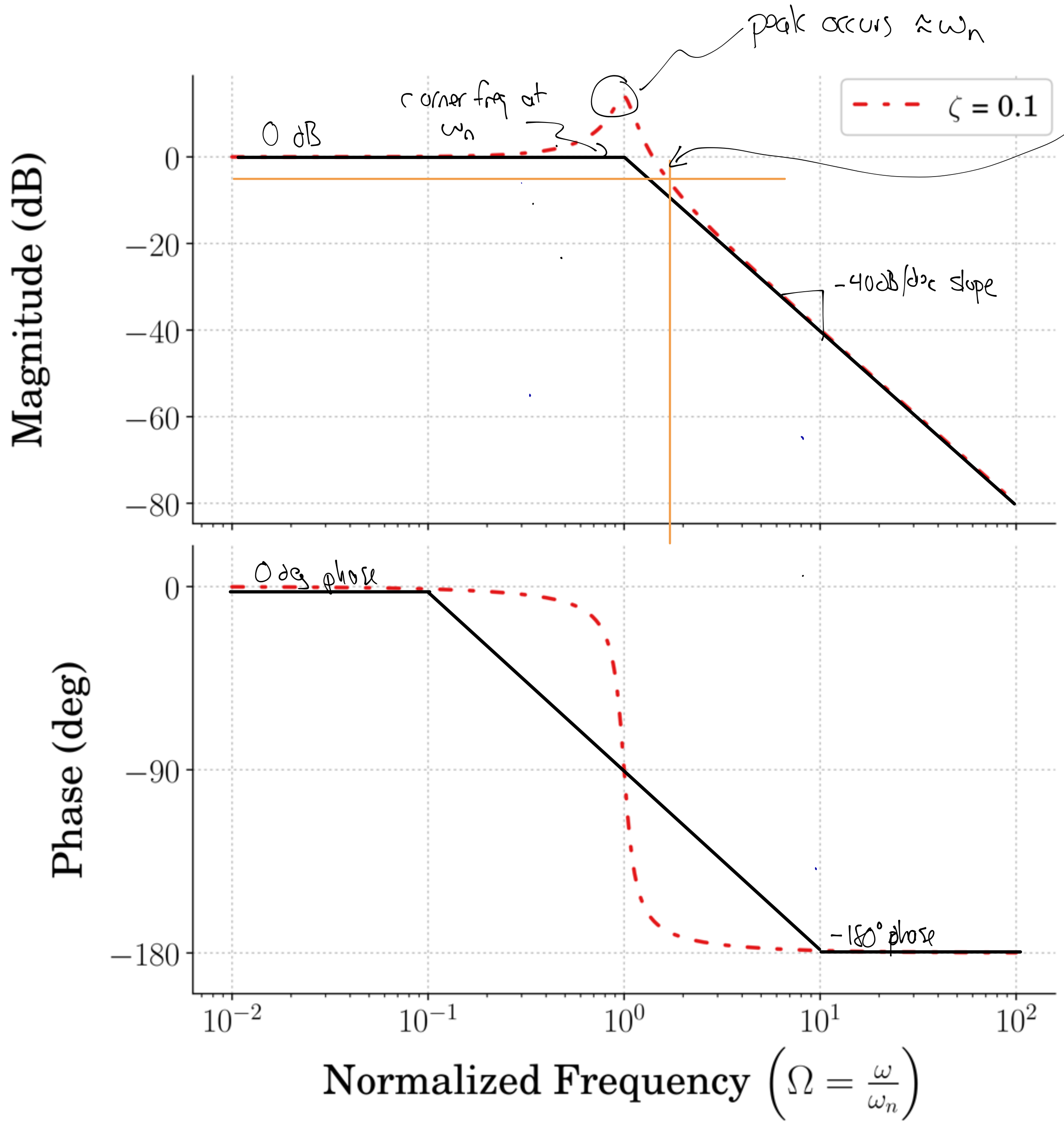
System Performance Analysis with Bode Plots



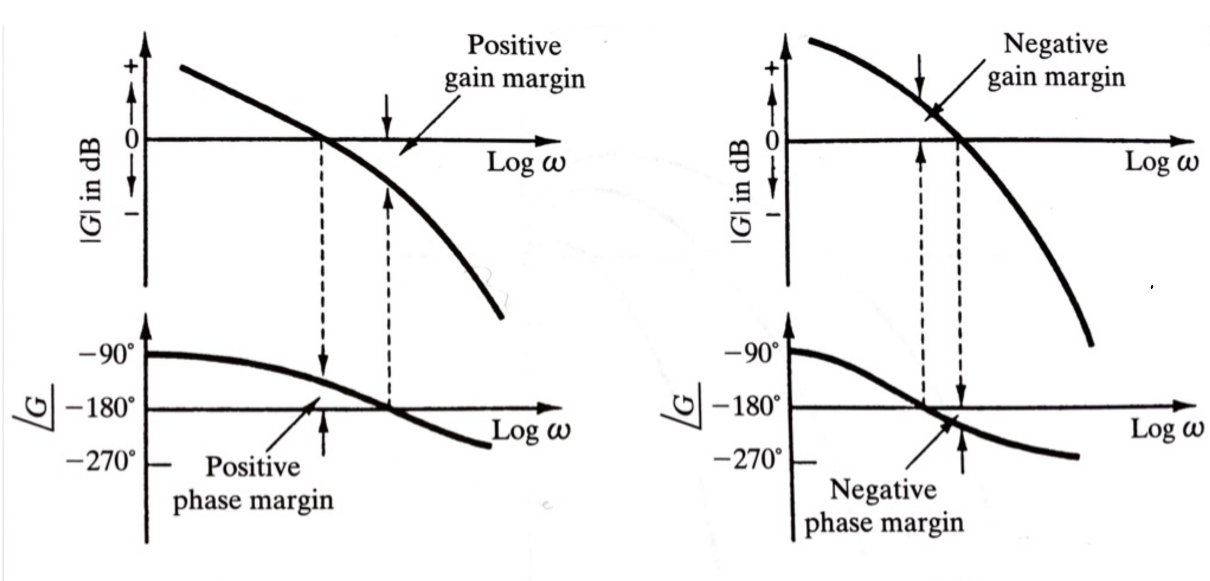
Bandwidth

Freq at which response has declined 3dB from ω≈0 value.

Measure of speed of response.

Gain Margin — Amount below 0dB when phase crosses -180deg

Phase Margin — Amount above -180deg when gain crosses 0dB



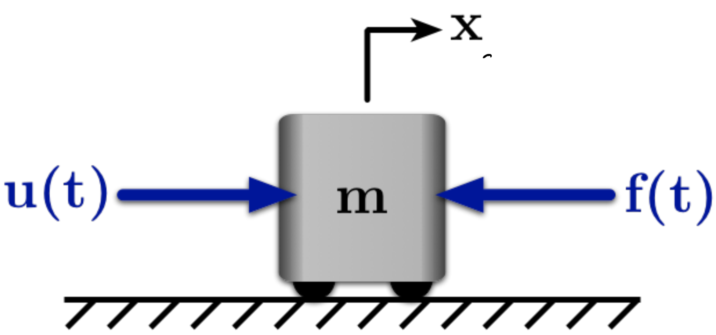
Stable system — Unstable system

Gain margin also shows how much gain can change before becoming stable/unstable

(How much to increase if stable or how much to decrease if unstable)

From the 1st Day of Class

Let's look again at the model



Found $m\ddot{x} = u - \cancel{f}$

For our PD controller

$$u(t) = k_p(x_d - x) + k_d(\dot{x}_d - \dot{x})$$

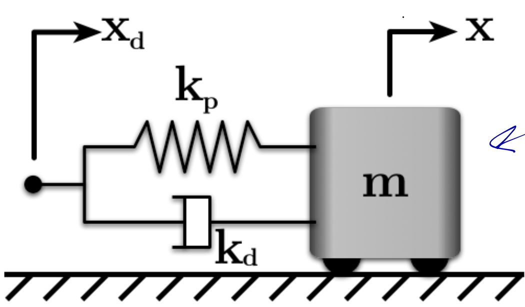
If $f(t) = 0$

$$m\ddot{x} = k_p(x_d - x) + k_d(\dot{x}_d - \dot{x})$$

Q: Does that equation look familiar?

Q: What mechanical element produces force prop. to a difference in position? ← linear spring

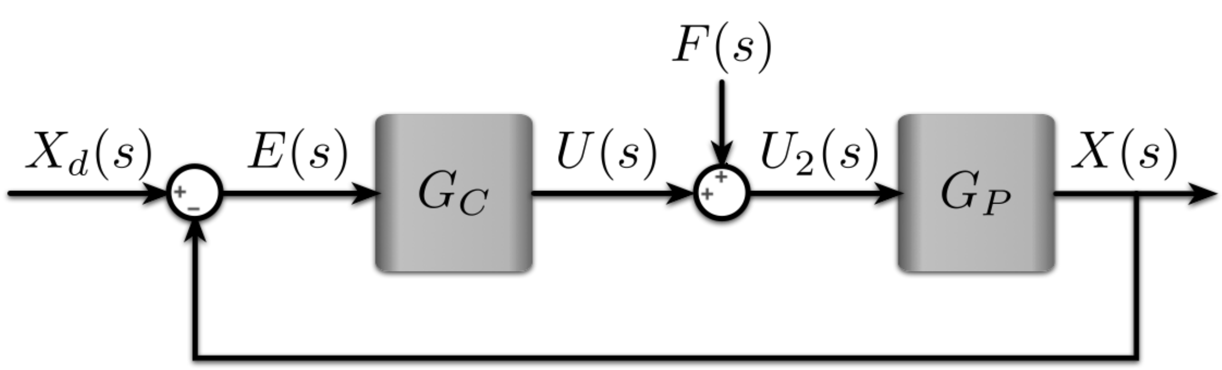
Q: What mechanical element produces force prop. to the rate of change of its length? ← viscous damper



Our PD controller is mathematically equivalent to this system!!!

Intuitively, you can think of a PD controller acting as a virtual spring-damper between the current state of the system and the desired state.

Q: What is the block diagram of this system?



Q: What is the TF from X_d to X, with $f(t) = 0$

$$m\ddot{x} + k_d\dot{x} + k_p x = k_d\dot{x}_d + k_p x \rightarrow (ms^2 + k_d s + k_p)X(s) = (k_d s + k_p)X_d(s)$$

$$\frac{X}{X_d} = \frac{k_d s + k_p}{ms^2 + k_d s + k_p} \rightarrow \text{divide by } m = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 2\zeta\omega_n = \frac{k_d}{m} \qquad \omega_n^2 = \frac{k_p}{m}$$

Q: What is the TF from F to X with $x_d(t) = 0$

$$m\ddot{x} = k_p(\cancel{x_d}^{\,0} - x) + k_d(\cancel{\dot{x}_d}^{\,0} - \dot{x}) - f$$

$$m\ddot{x} + k_d\dot{x} + k_p x = f \rightarrow (ms^2 + k_d s + k_p)X = F$$

$$\frac{X}{F} = \frac{1}{ms^2 + k_d s + k_p} \rightarrow \text{divide by } m \rightarrow \frac{X}{F} = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$2\zeta\omega_n = \frac{k_d}{s} \qquad \omega_n^2 = \frac{k_p}{m}$$

Q: If we want $X = 0$ for all time ($x_d(t) = 0$) how should we select k_p and k_d?
steady state

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Look at step input in $f(t)$, $F(s) = \frac{1}{s}$

$$= \lim_{s\to 0} \cancel{s}\left[\frac{1}{ms^2 + k_d s + k_p}\left(\frac{1}{\cancel{s}}\right)\right] = \lim_{s\to 0} \frac{1}{ms^2 + k_d s + k_p} = \frac{1}{k_p} \leftarrow \text{we need a large } k_p$$

Q: Can X_{ss} ever be zero for this case?

No $k_p \rightarrow \infty$ for that to happen

Q: How can we select gains $k_p + k_d$?

1) It's a 2^{nd} order system, so we can work directly from the perf measures

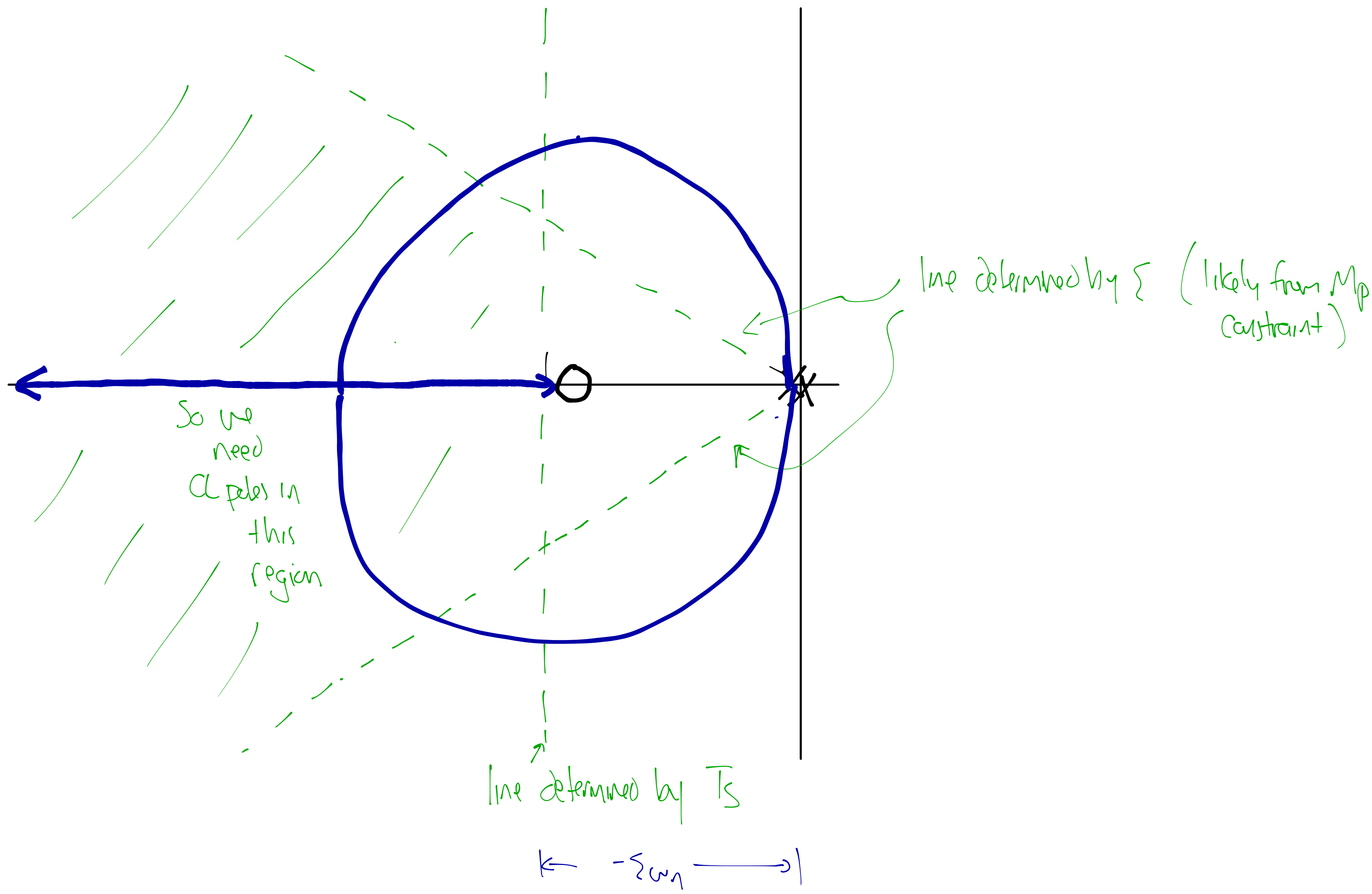
$$M_p = 100\% \times \exp\left[\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right]$$

$$T_p = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

$\vdots$

These also map out regions on the root locus



The CL TF for this system

$$G_cG_p = (k_d s + k_p)\left(\frac{1}{ms^2}\right) = \frac{k_d s + k_p}{ms^2} \rightarrow \frac{k_p\left(\frac{k_d}{k_p}s + 1\right)}{ms^2}$$

fix this ratio to start

zero at $-\frac{k_p}{k_d}$

Double pole at 0

Asymptotes:

$$\sigma_A = \frac{\sum p_i - \sum z_i}{n-m} = \frac{0 - \frac{k_p}{k_d}}{2-1} = -k_p/k_d$$

$$\phi_A = \left(\frac{2k+1}{n-m}\right)180^\circ \quad k=0 \rightarrow \phi_A = 180^\circ$$

Q: What can we say about the freq. resp on this system?

Sketch the Bode Plot of closed-loop system

$$\frac{X}{X_d} = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 2\zeta\omega_n = \frac{K_d}{m} \qquad \omega_n^2 = \frac{K_p}{m}$$

zero at $\frac{-\omega_n}{2\zeta} \rightarrow$ use 10 here

complex conjugate poles $\rightarrow \omega_n$