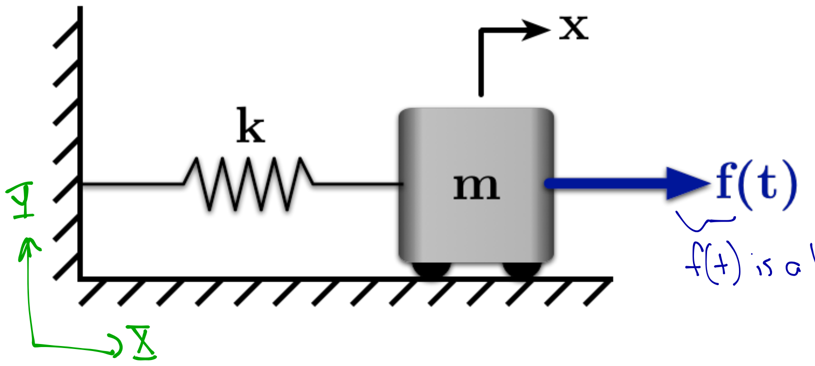
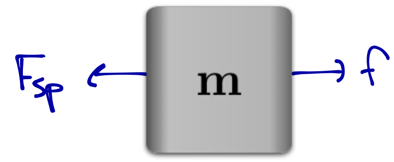
# Direct Force Excitation with No Damping




$\underbrace{f(t)}$ is a harmonic force

FBD




$m\ddot{x} = -F_{sp} + f = -kx + f$

$m\ddot{x} + kx = f$ or (divide by m) $\ddot{x} + \omega_n^2 x = \frac{1}{m} f$

Assume $f(t) = \bar{f} \sin \omega t \longrightarrow$ expect the solution to be of form $x(t) = \bar{X} \sin \omega t$

Now, just plug this assumed solution into the eq. of motion and solve for $\bar{X}$

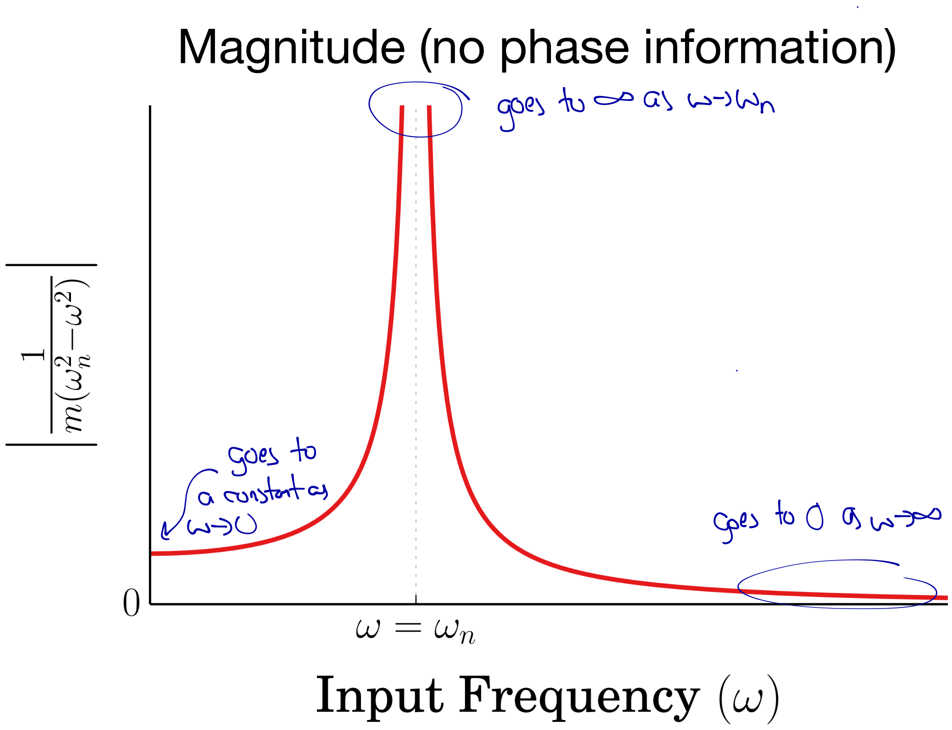
$$\left(-\omega^2 \bar{X} \cancel{\sin \omega t}\right) + \omega_n^2 \left(\bar{X} \cancel{\sin \omega t}\right) = \frac{1}{m} \bar{f} \cancel{\sin \omega t}$$

$$\left(\omega_n^2 - \omega^2\right) \bar{X} = \frac{1}{m} \bar{f}$$

$$\bar{X} = \frac{\bar{f}}{m} \left(\frac{1}{\omega_n^2 - \omega^2}\right) \longleftarrow$$ This is the amplitude of vibration of m

So, $x(t) = \frac{1}{m(\omega_n^2 - \omega^2)} \bar{f} \sin \omega t = \underbrace{\left[\frac{1}{m(\omega_n^2 - \omega^2)}\right]} f(t)$

Let's plot this term. (Aside: This is actually just the TF for this system)



## Normalization

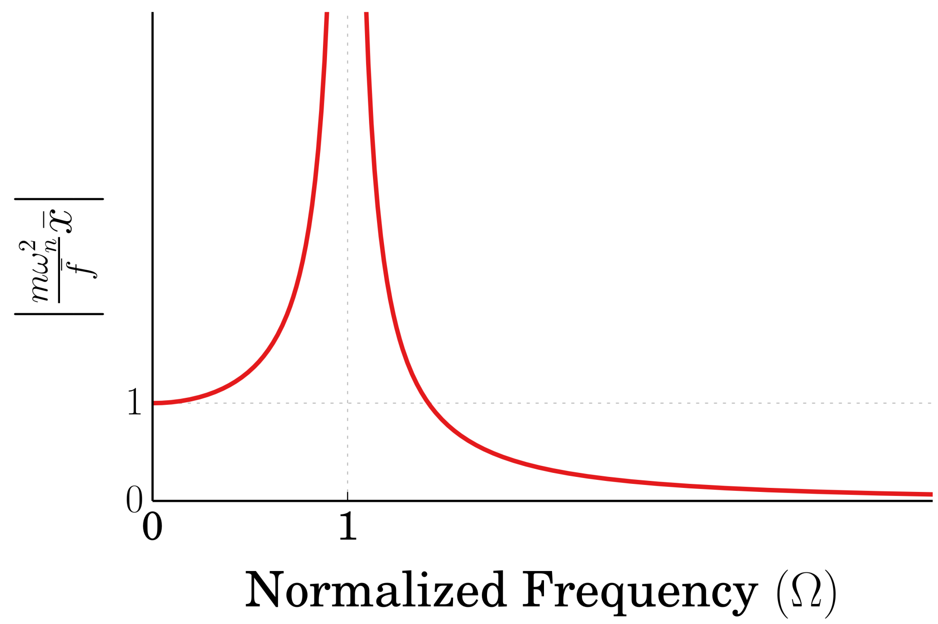
We can also normalize this by using the nondimensional frequency, $\Omega$ .

To even further normalize, we can scale the amplitude such that it is 1 when the excitation frequency is 0.

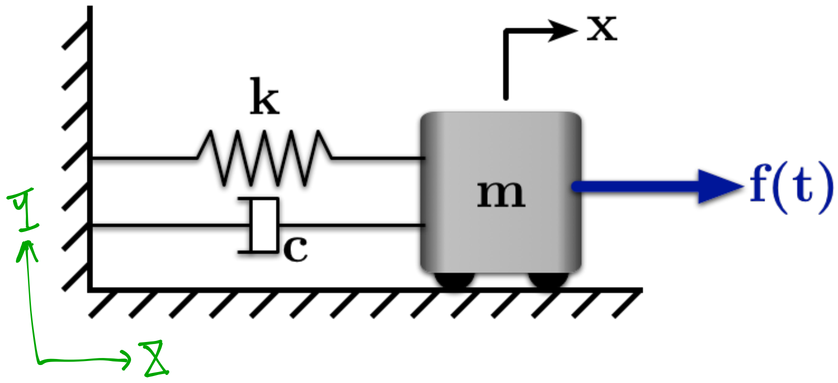
$$x(t) = \frac{1}{m(\omega_n^2 - \omega^2)} \bar{f} \sin \omega t \longrightarrow \frac{1/\omega_n^2}{m\left(\frac{\omega_n^2}{\omega_n^2} - \frac{\omega^2}{\omega_n^2}\right)} \bar{f} \sin \omega t = \underbrace{\frac{1}{m\omega_n^2(1-\Omega^2)} \bar{f}}_{\bar{x} = \frac{\bar{f}}{m\omega_n^2(1-\Omega^2)}} \sin \omega t$$

Plot $\frac{m\omega_n^2}{\bar{f}} \bar{x}$ vs. $\Omega$

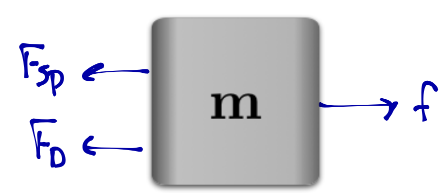
Magnitude (no phase information)



# Forced Vibration with Viscous Damping




FBD




Damped equation of motion

$$m\ddot{x} = f - F_{sp} - F_D = f - kx - c\dot{x}$$

$$m\ddot{x} + c\dot{x} + kx = f \longrightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m} f \longleftarrow \text{Assume } f(t) = \bar{f}\sin\omega t$$

Like the undamped case, assume a solution of:

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

Plug into the eq. of motion and group sin and cos terms:

$$(-\omega^2 a + 2\zeta\omega_n\omega b + \omega_n^2 a)\cos\omega t + (-\omega^2 b - 2\zeta\omega_n\omega a + \omega_n^2 b)\sin\omega t = \frac{1}{m}\bar{f}\sin\omega t$$

Match sine and cosine terms:

$$\cos \rightarrow \quad -\omega^2 a + 2\zeta\omega_n\omega b + \omega_n^2 a = 0$$

$$\sin \rightarrow \quad -\omega^2 b - 2\zeta\omega_n\omega a + \omega_n^2 b = \bar{f}/m$$

Note:

1) With damping, the cosine term remains
2) $b = 0$ iff $\omega = \omega_n$ and $a$ is finite at $\omega = \omega_n$ ← The response is always finite

BUT this solution is ugly

Q: How else could we write the response?

$$x(t) = |\bar{x}|\cos(\omega t - \phi) \quad \text{or} \quad x(t) = |\bar{x}|\sin(\omega t - \phi)$$

We can use either, but usually best to match the form of the input. Ex) If $f(t) = \bar{f}\sin\omega t$, then choose $x(t) = |\bar{x}|\sin(\omega t - \phi)$

# Forced Vibration with Viscous Damping (cont.)

So, convert the "ugly" solution to this form

⋮ (Lot's of math)

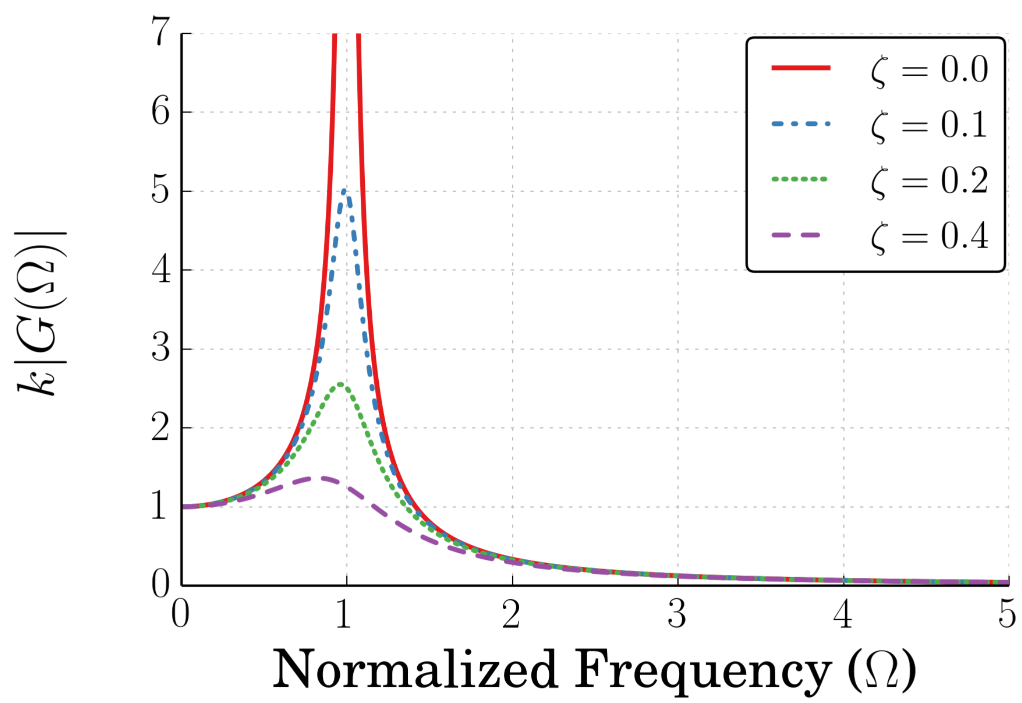
$$|\bar{X}| = \frac{\bar{f}}{m}\frac{1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)$$

We can get the transfer function form of this (divide by $\bar{f}$)

$$|G(\omega)| = \frac{|\bar{X}|}{|\bar{f}|} = \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}$$

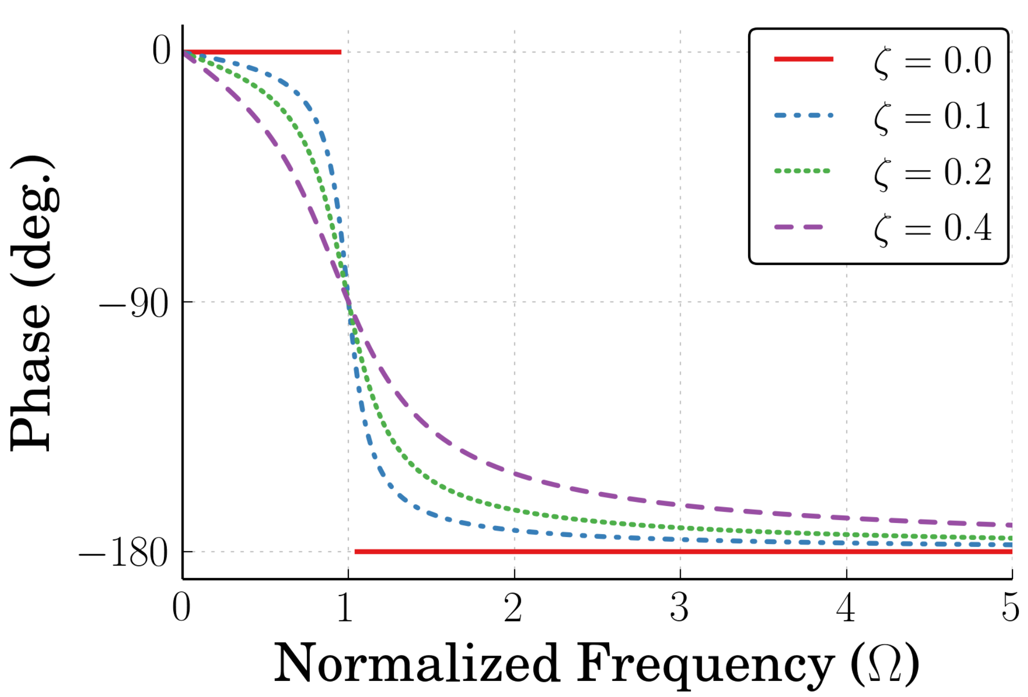
We can also normalize by $\Omega = \frac{\omega}{\omega_n}$ $\rightarrow$ $|G(\Omega)| = \frac{1}{k\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}}$ and $\phi = \tan^{-1}\left(\frac{2\zeta\Omega}{1-\Omega^2}\right)$

plot $k|G(\Omega)|$ to normalize the amplitude




Notice that damping limits the peak of the response at the natural frequency.

The trends as the input frequency goes to zero or infinity are the same for undamped and damped cases.




Notice that the damped responses transition from 0 to -180 deg phase shift.

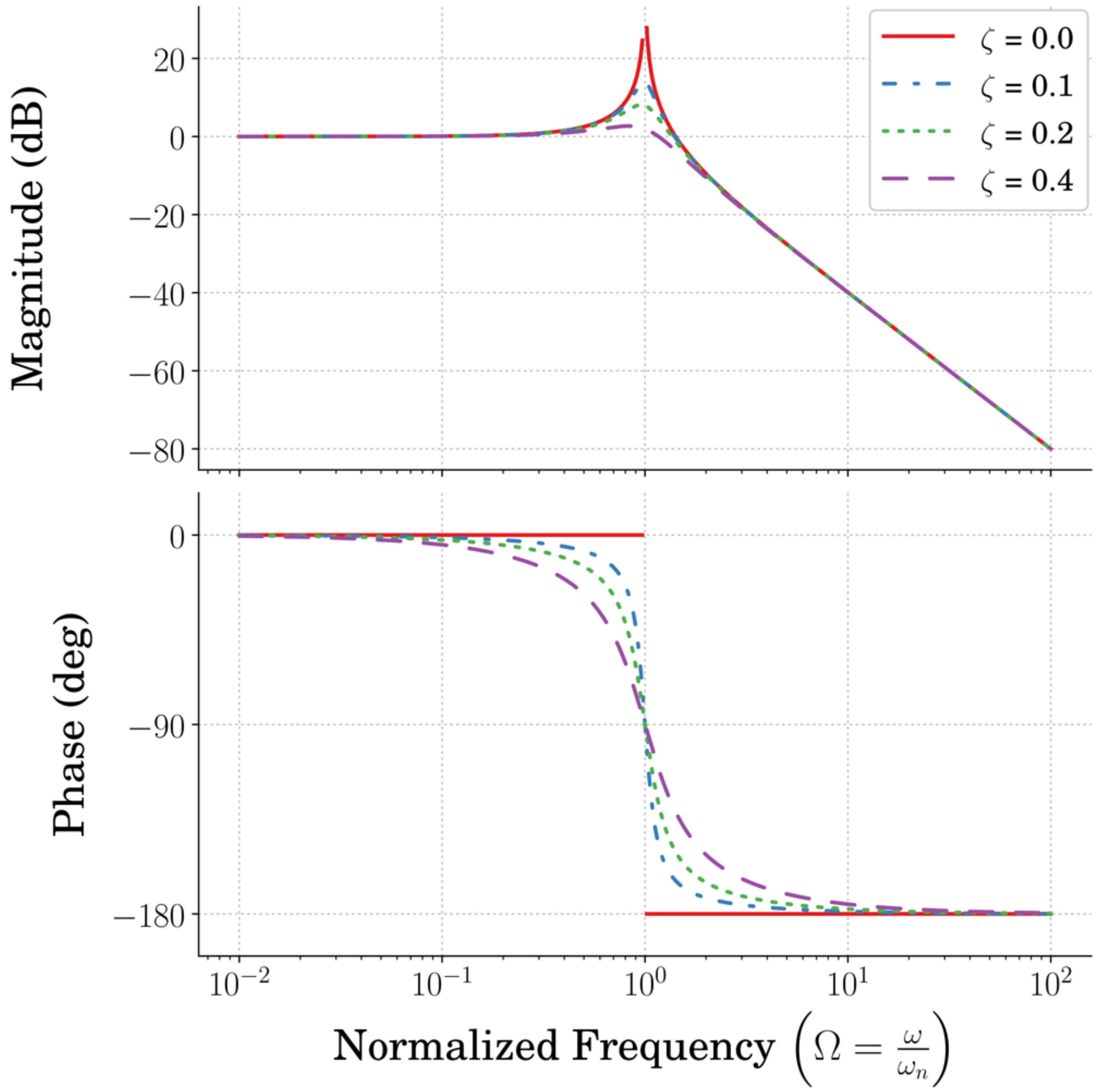
# Bode Plots (Sec. 8.2)

Very similar to plots in previous section of notes, but with different scaling

Freq (horiz axis) is plotted on a log scale

Magnitude is written in dB → $|G_p| \rightarrow 20\log_{10}(|G_p|)$

So, the freq resp from the previous page becomes:

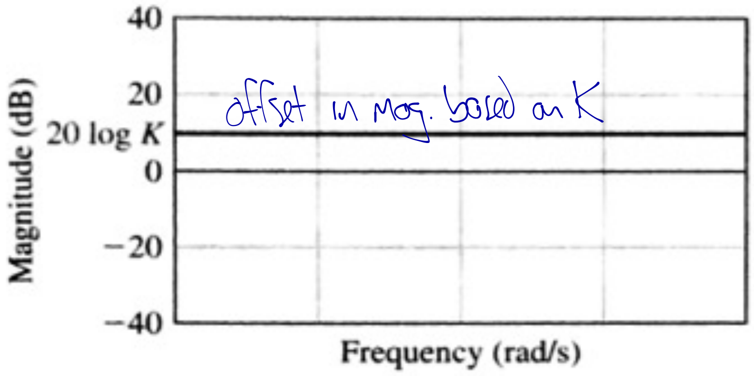
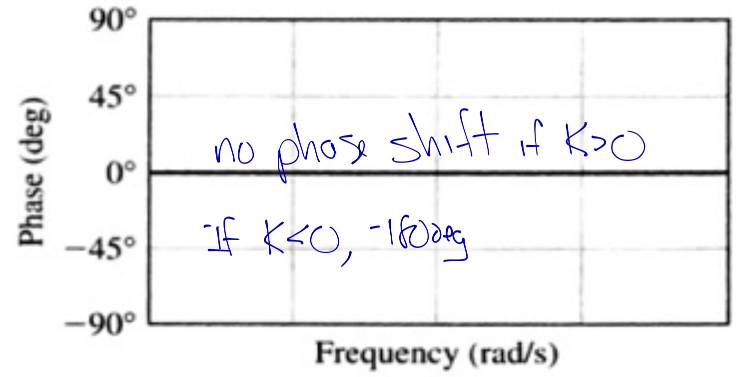
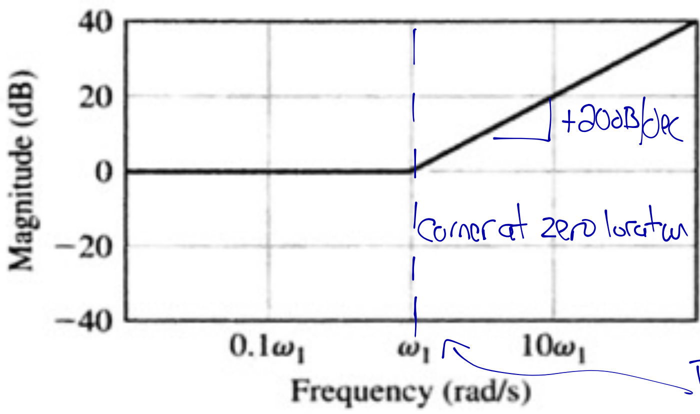
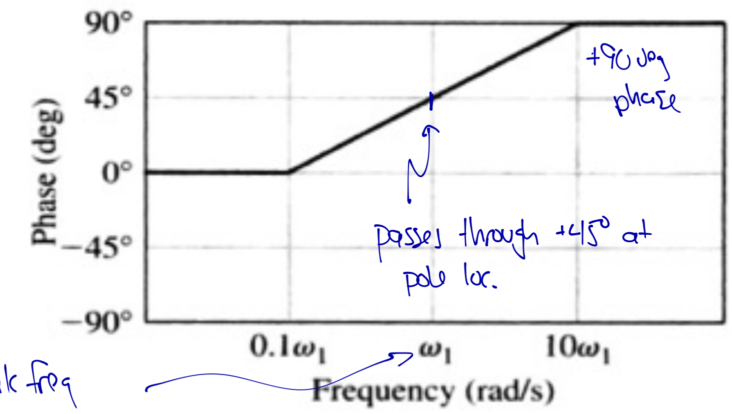
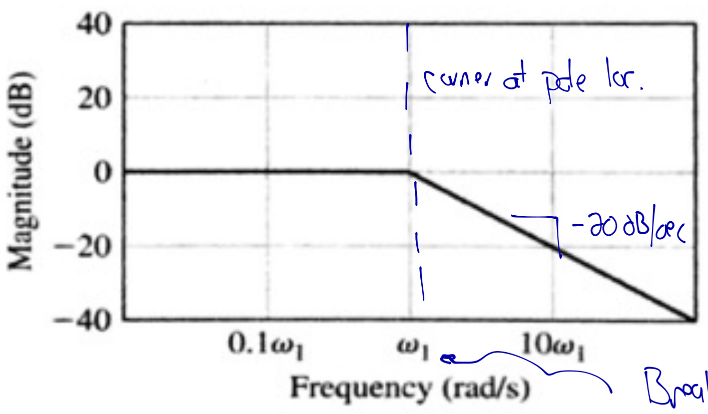
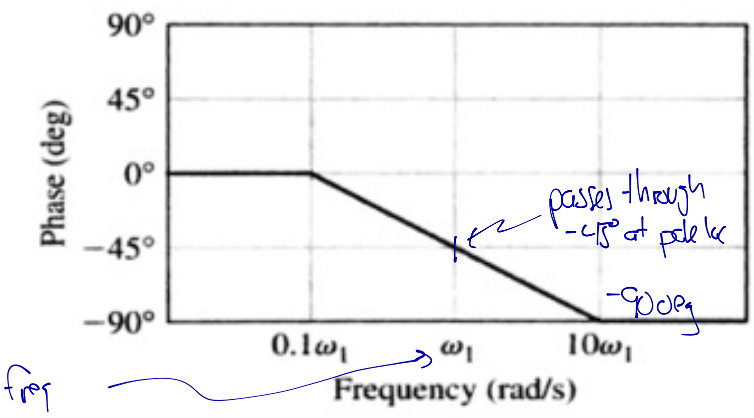
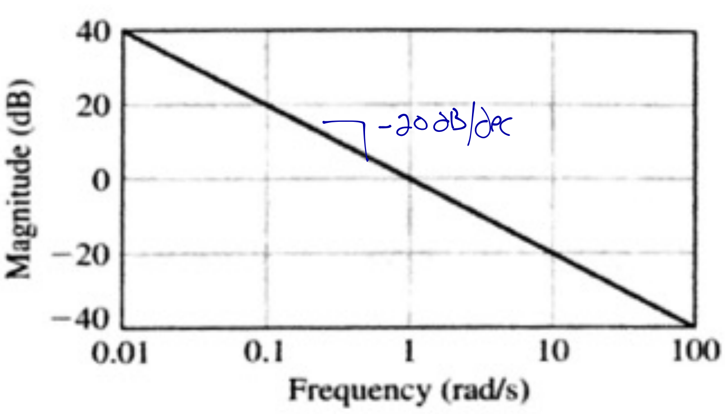
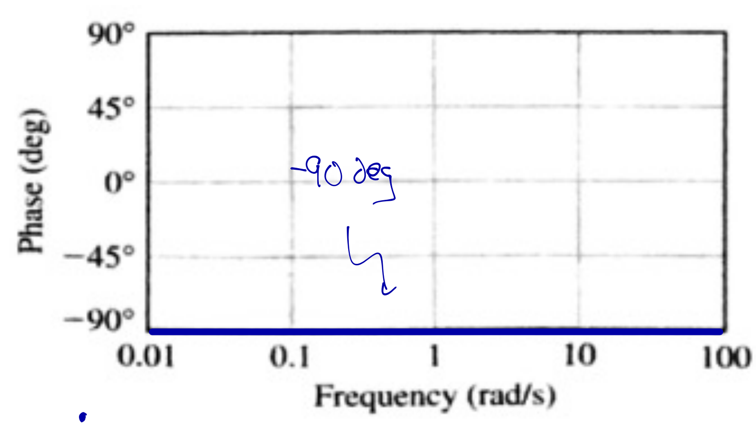
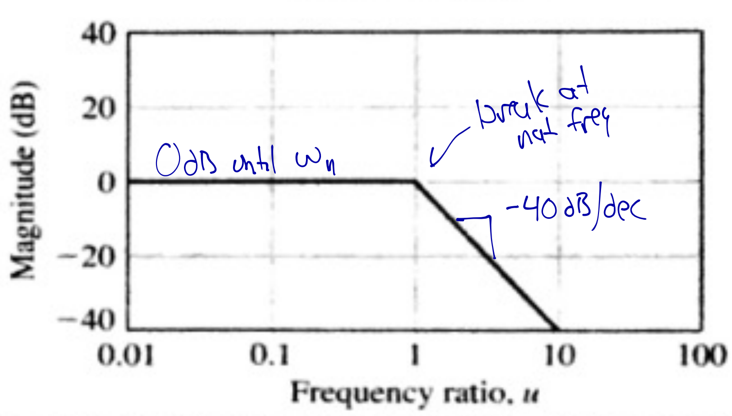
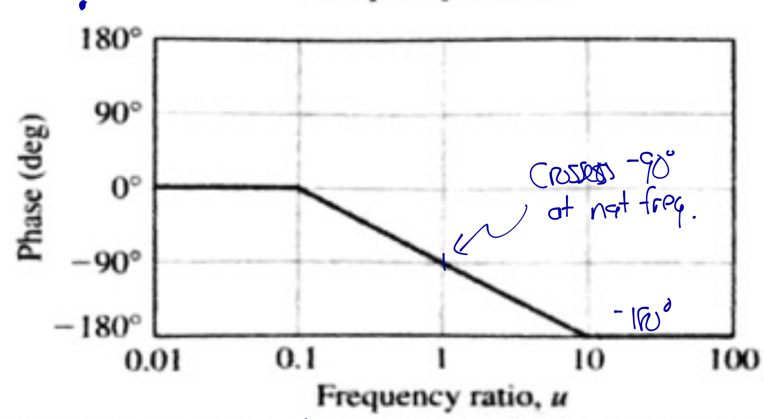



What's nice about this form is that we can easily:

- sketch an approx of those curves directly from the TF
- Using the same approx, go from an experimental freq response to a TF

# Bode Plots (cont.)

**Table 8.1 Asymptotic Curves for Basic Terms of a Transfer Function**

| Term | Magnitude $20 \log_{10}\lvert G(j\omega)\rvert$ | Phase $\phi(\omega)$ |
|---|---|---|
| 1. Gain, $G(j\omega) = K$ |  |  |
| 2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$ |  |  |
| 3. Pole, $G(j\omega) = (1 + j\omega/\omega_1)^{-1}$ |  |  |
| 4. Pole at the origin, $G(j\omega) = 1/j\omega$ |  |  |
| 5. Two complex poles, $0.1 < \zeta < 1$, $G(j\omega) = (1 + j2\zeta u - u^2)^{-1}$, $u = \omega/\omega_n$ |  |  |

notice the change in horiz scale

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# Bode Plots (cont.)

Let's look back to the previous example

$$G_p = \frac{X}{F} = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ ← complex poles at $-\zeta\omega_n \pm i\omega_d$ ← let's plot for $\omega_n = 1$