

Chapter 6 - The Stability of Linear (Feedback) Systems

Stable System - given a bounded (i.e. not $\rightarrow \infty$) input, the response is bounded

- system poles all lie in Left half plane (negative real parts)
- eigenvalues of A matrix all have negative real part

If poles are exactly on imaginary axis (real part = 0), then the system:

- has one undamped mode
- can have an unbounded response for a bounded input iff its freq exactly matches the mode whose poles lie on the imaginary axis
- will have a bounded response in all other cases
- is called marginally stable

The Routh-Hurwitz Stability Criterion (Sec. 6.2)

A method to determine the stability of a system without having to calculate all its roots

This was very useful before we could easily find roots computationally.

Its usefulness is much less now, given computational power.

Chapter 7 - The Root Locus Method

Tool to visualize how the pole(s) move around the s-plane as a parameter is varied.

Can (of course) be calculated exactly, but we'll also learn to sketch approximate loci for quick/approx system analysis.

The Root Locus Concept (Sec. 7.2)

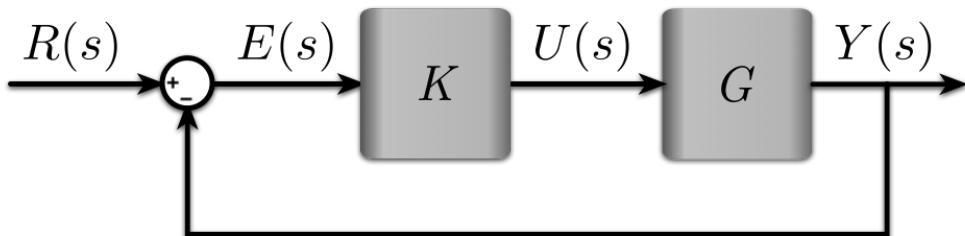
Consider closed-loop TF, $T(s)$:

$$T(s) = \frac{Y}{R} = \frac{p(s)}{q(s)}$$

$p(s)$ - polynomial in s rep. numerator

$q(s)$ - polynomial in s rep. denominator

roots of this polynomials are sys. poles



For this system:

$$T = \frac{Y}{R} = \frac{KG}{1+KG} \xrightarrow{\text{char. eq}} 1+KG=0 \quad \text{where } K \text{ is variable}$$

$$0 \leq K < \infty$$

$1+KG=0$ is an equation in s -domain $\leftarrow KG = -1$

s is a complex variable, so we can write in polar form $\rightarrow \alpha + i\beta = |\alpha + i\beta| \angle (\alpha + i\beta)$

So, $KG = -1 \rightarrow |KG| / KG = 1 + j0 : 1 + i0 \leftarrow$ Notice that

we're once again using the OLTF for into about CL resp.

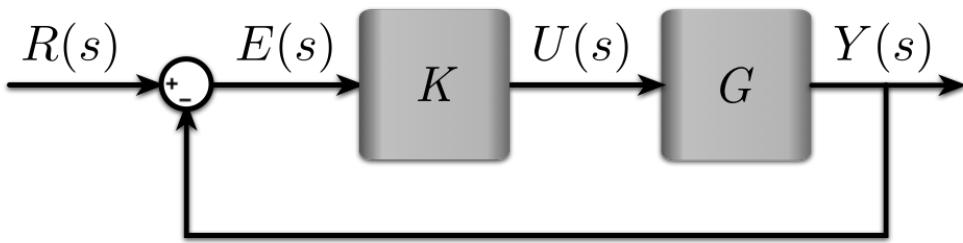
This can then be written as

$$|KG| = 1 \quad \text{and} \quad \angle KG = 180^\circ + k360^\circ \quad \text{where } k = 0, \pm 1, \pm 2, \dots$$

All points found solving these 2 eq. as K varies define the root locus.

The Root Locus Concept (cont.)

Consider this system: where $G(s) = \frac{1}{s(s+2)}$



Q: What is the closed-loop transfer function?

$$\frac{Y}{R} = \frac{KG}{1+KG} = \frac{\frac{K}{s(s+2)}}{1 + \frac{K}{s(s+2)}} = \frac{K}{s(s+2) + K} = \frac{K}{s^2 + 2s + K}$$

So the characteristic equation is:

$$\Delta(s) = s^2 + 2s + K = 0$$

which we can (and often do) write as

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

To find the locus of roots for this system as K is varied:

$$|KG| = \left| \frac{K}{s(s+2)} \right| = 1 \quad \text{and} \quad \angle KG = \pm 180^\circ, \pm 540^\circ, \dots$$

The Root Locus Procedure (Sec. 7.3)

- i) Prepare the system TF and p_2 -map for the sketch
- ii) Write the char. eq. such that parameter of interest, K , is:

$$1 + KG = 0$$

- iii) Factor G into poles and zeros:

$$1 + K \frac{\prod_{i=1}^M (s+z_i)}{\prod_{j=1}^N (s+p_j)} = 0$$

- iv) Plot open-loop poles (as x) and zeros (as o) on the s-plane

We'll see that when we sketch the locus, as the gain is increased the closed-loop poles move from open-loop poles toward open-loop zeros

- v) Determine the number of separate loci, SL

$$SL = n \quad \text{where } n = \# \text{ of finite poles} \quad \text{and when } n \geq M \quad (\text{more poles than zeros})$$

$M = \# \text{ of finite zeros}$

- vi) Locate loci components on the real axis. Locus is left of odd number poles and zeros

- vii) Loci go to zeros at infinity along asymptotes

sounds strange but
will discuss more
soon

centered at:

$$\sigma_A = \frac{\sum (-p_j) - \sum (-z_i)}{n-M}$$

at angle

$$\phi_A = \left(\frac{2k+1}{n-M} \right) 180^\circ \quad k = 0, 1, \dots (n-M-1)$$

- viii) Determine where the loci cross the imaginary axis

The Root Locus Procedure (cont.)

5) Determine the breakaway points (There may be none.)

a) set $K = p(s)$

b) find roots of $\frac{dp(s)}{ds} = 0$ or use "graphical" method

c) Determine angle of departure from complex poles and angle of arrival at complex zeros

$$\angle G(s) : 180^\circ + k \cdot 360^\circ \text{ at } s = -p_j \text{ and } s = -z_i$$

→ Sketch the locus

Example 7.1

Given characteristic equation:

$$\textcircled{1} \quad \text{a) } 1 + G_C G_P = 1 + K \frac{2(s+2)}{s^2 + 4s}$$

This TF is "P" in book notation. We'll call it $G(s)$

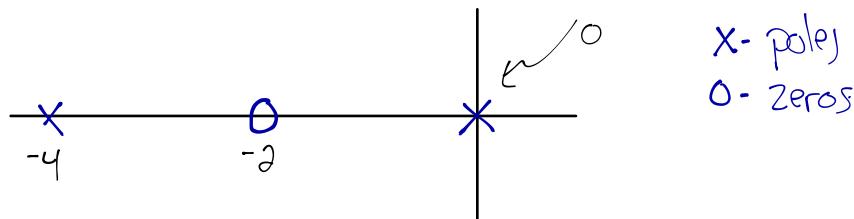
$$\text{So } G(s) = \frac{2(s+2)}{s^2 + 4s}$$

b) Rewrite as factored to find poles and zeros

$$1 + K \frac{2(s+2)}{s(s+4)} = 0$$

← zero at -2
← poles at 0 and -4

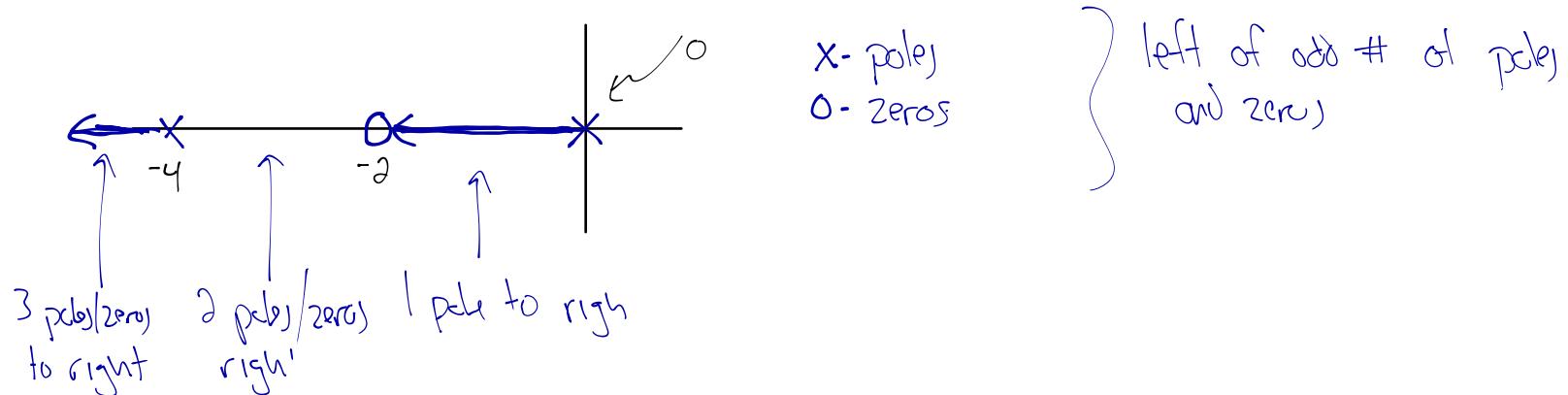
c) Plot poles and zeros on s-plane



d) Determine number of separate loci

here: $M=1$ (1 zero) and $N=2$ (2 poles) so will have 2 separate loops

② Locate the component(s) on the real axis

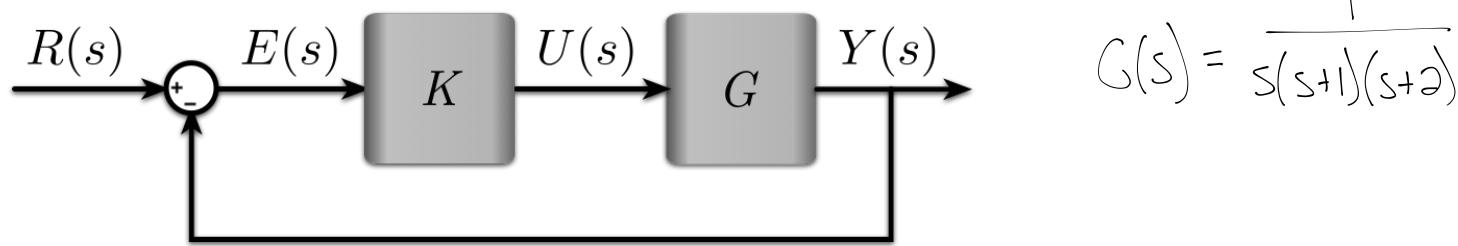


$$\textcircled{3} \quad \sigma_A = \frac{\sum (-p_i) - \sum (-z_i)}{n-m} = \frac{(0-4) - (-2)}{2-1} = -2$$

$$\phi_A = \left(\frac{2k+1}{2} \right) 180^\circ \leftarrow \text{se segments are all on real axis (we're done sketching)}$$

We could also work through remaining steps if we were unsure of this

Example



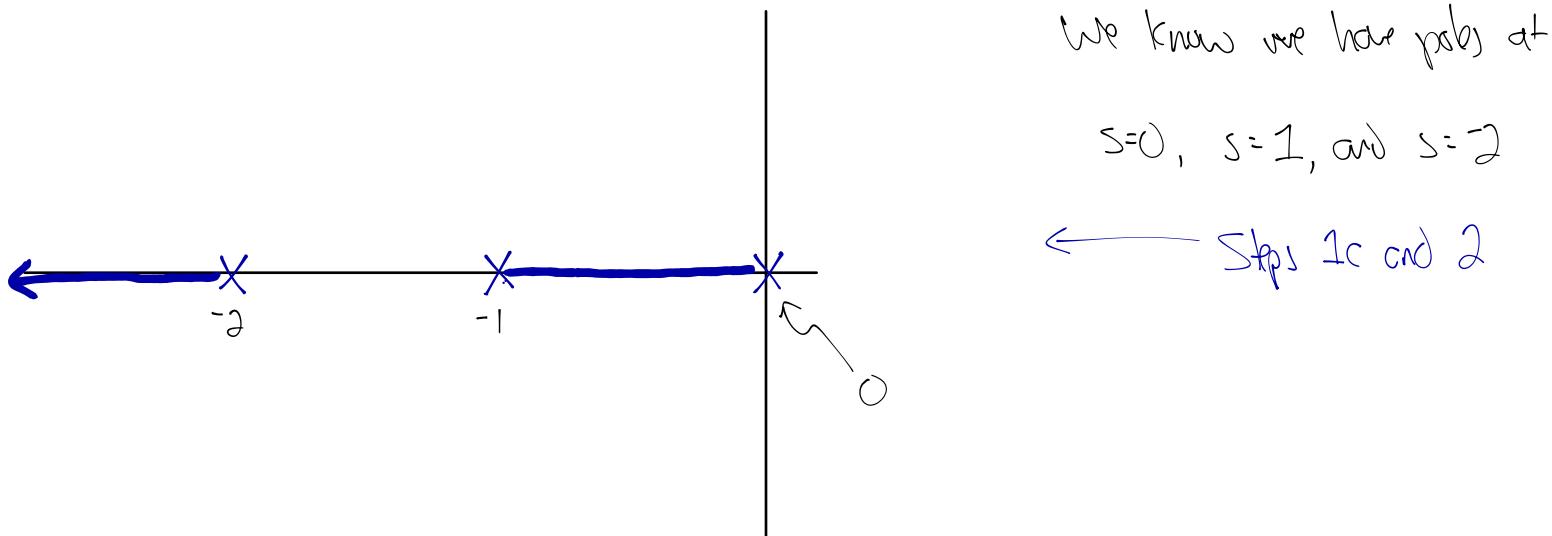
The closed-loop TF = $\frac{KG}{1+KG}$ so the char. eq is $1+KG=0$ ← steps 1a-b

So the magnitude condition is:

$$\left| \frac{KG}{s(s+1)(s+2)} \right| = 1$$

And the angle condition is

$$\angle \frac{KG}{s(s+1)(s+2)} = \pm 180^\circ, \pm 540^\circ, \dots$$



We know there are 0 zeros and 3 poles so $SL=n=3$ ← Step 1d

Example (cont.)

Now find asymptotes of loci

Centred at:

$$\sigma_A = \frac{\sum(-p_j) - \sum(-z_i)}{n-m} = \frac{(-1-2)}{3} = -1$$

at angle

$$\phi_A = \left(\frac{2k+1}{n-m} \right) 180^\circ \quad k = 0, 1, \dots (n-M-1)$$

$$= \left(\frac{2k+1}{3} \right) 180^\circ \quad \leftarrow \text{asymptotes at } 180^\circ \text{ and } 60^\circ$$

Step 3

We now want to find the breakaway point(s) \leftarrow where the locus leaves the real axis \leftarrow Step 5

Write char eq. a)

$$1 + \frac{K}{s(s+1)(s+2)} = 0 \rightarrow K = -s(s+1)(s+2) = -s(s^2+3s+2) = -s^3 - 3s^2 - 2s$$

$$K = -s^3 - 3s^2 - 2s$$

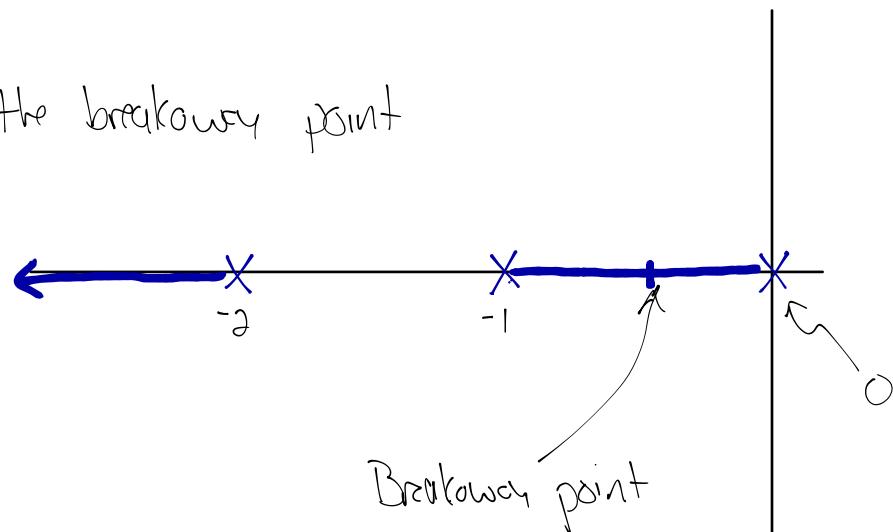
Now solve $\frac{dK}{ds} = 0$ to find potential breakaway points

$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0 \rightarrow -1(3s^2 + 6s + 2) = 0$$

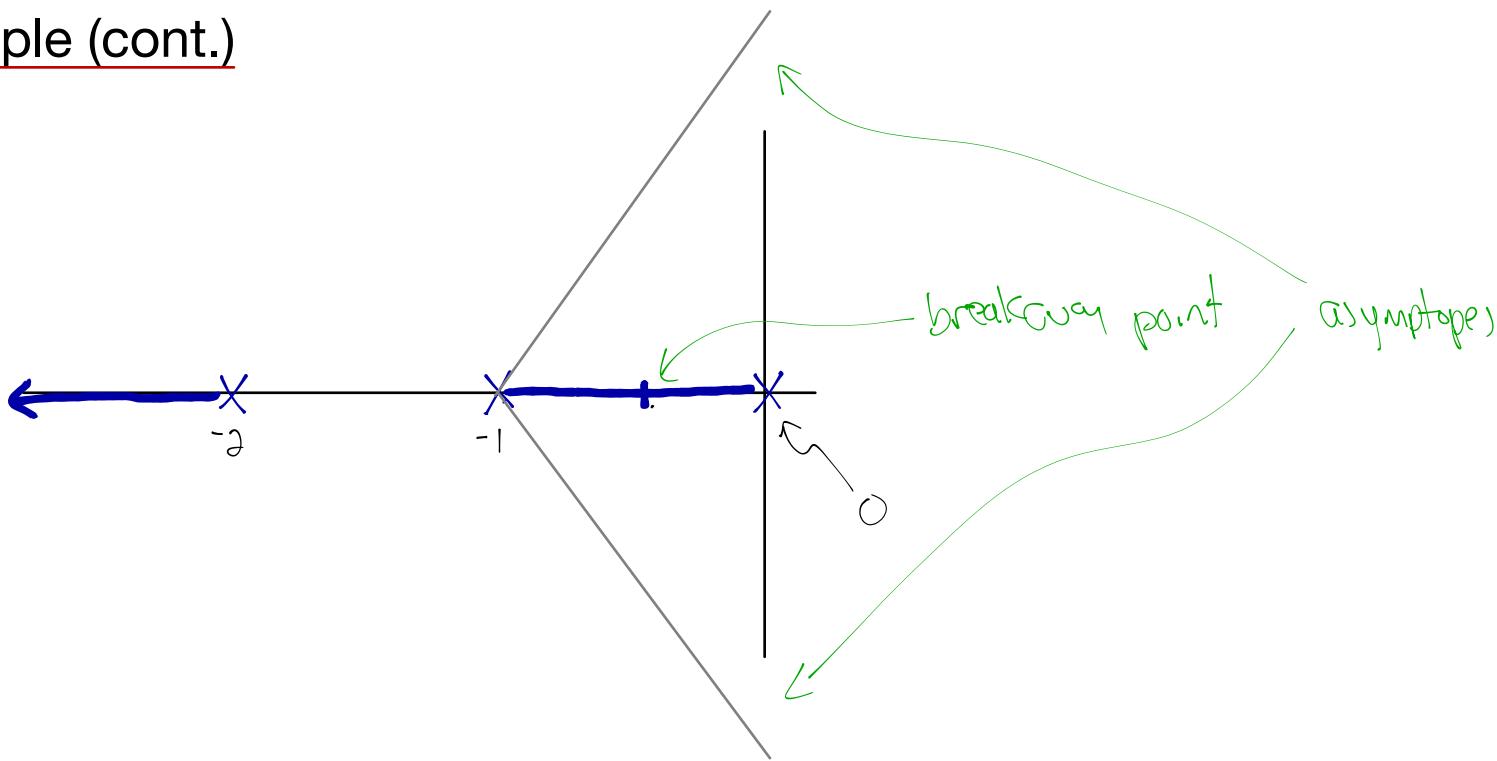
$$s = \frac{-6 \pm \sqrt{36 - 24}}{6} = \frac{-6 \pm \sqrt{12}}{6} = \frac{-6 \pm 2\sqrt{3}}{6} = -1 \pm \frac{1}{3}\sqrt{3} \rightarrow s = -0.4226 \quad s = -1.5774$$

Q: Which of those lie on the locus?

only $s = -0.4226 \leftarrow$ so it's the breakaway point



Example (cont.)



We also should find where the loci crosses the imaginary axis.

Q: Why is this important?

The system goes unstable at this point

One way

plug in (iw) for s and solve:

$$s^3 + 3s^2 + 2s + K = 0 \rightarrow (iw)^3 + 3(iw)^2 + 2(iw) + K = 0$$

$$(K - 3\omega^2) + i(2\omega - \omega^3) = 0 \leftarrow \text{Find } K \text{ and } \omega \text{ that cause this to be true.}$$

Q: What has to happen for that to be true?

Both real and imaginary components = 0

$$K - 3\omega^2 = 0 \quad \text{and} \quad 2\omega - \omega^3 = 0$$

$$K - 3(\omega) = 0 \rightarrow K = 0 \quad \xrightarrow{\text{sub}} \quad \omega(2 - \omega^2) = 0 \leftarrow \omega = 0 \quad \text{or} \quad \omega = \pm\sqrt{2}$$

$$K - 3(2) = 0 \rightarrow K = 6 \quad \omega = 0, K = 0 \quad \text{and} \quad \omega = \pm\sqrt{2}, K = 6$$

Example (cont.)

