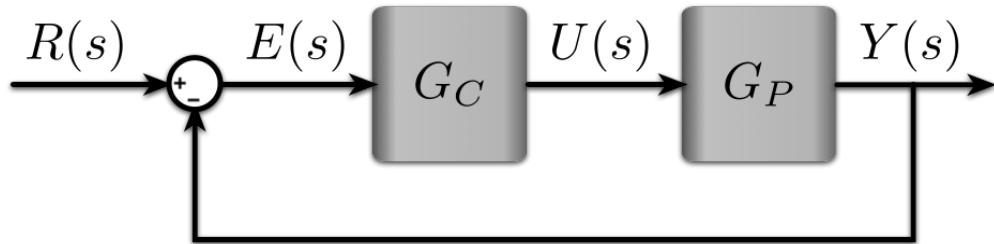


Steady-state Error of Systems (Sec. 5.6)

We already seen how to do this using the final value theorem. We'll now look at the error to over 3 "standard" test inputs (step, ramp, accel/parabola).

For our "normal" unity-feedback system



is

$$E(s) = \frac{1}{1 + G_C G_P} R(s)$$

Step Input

$$R(s) = \frac{A}{s} \rightarrow E(s) = \frac{A}{s(1 + G_C G_P)}$$

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{A}{1 + G_C G_P} = \frac{A}{1 + \lim_{s \rightarrow 0} G_C G_P}$$

So the open-loop TF determines the steady-state error!!!

Write a general form of this TF as:

$$G_C G_P = \frac{K \prod_{i=1}^M (s + z_i)}{\prod_{k=1}^N (s + p_k)}$$

\prod ← product
 $z_i \neq 0$
 $p_k \neq 0$
 $1 \leq i \leq M$
 $i \leq k \leq N$
 more poles than zeros

This term (a factor of "pure" s's in the den.)
is called a pure integrator

✓ For this system, we have N pure integrators

If $N > 0$, then $\lim_{s \rightarrow 0} G_C G_P \rightarrow \infty$ and $E_{ss} \rightarrow 0$

↑ The number of pure integrators defines the type number of the system

Steady-state Error of Systems (cont.)

For a Type Zero system ($N=0$), the steady-state error is

$$e_{ss} = \frac{A}{1 + G_c(s)G_p(s)} = \frac{A}{1 + K_p}$$

where $K_p = \lim_{s \rightarrow 0} G_c G_p \leftarrow \text{Position Error Constant}$

Ramp Input

$$R(s) = \frac{A}{s^2} \rightarrow E(s) = \frac{A}{s(1 + G_c G_p)}$$

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{A}{s(1 + G_c G_p)} = \lim_{s \rightarrow 0} \frac{A}{s + sG_c G_p} = \underbrace{\lim_{s \rightarrow 0} \frac{A}{s G_c G_p}}$$

So the error again depends on
the Type Number of the system
(and open-loop TF)

For $N=0$, $e_{ss} \rightarrow \infty$

For a Type 1 system:

$$G_c G_p = \frac{K \prod_{i=1}^m (s + z_i)}{\prod_{k=1}^n (s + p_k)} \rightarrow e_{ss} = \frac{A}{K(\prod z_i / \prod p_k)} = \frac{A}{K_V}$$

$$K_V = \lim_{s \rightarrow 0} s G_c G_p$$

$K_V \equiv$ Velocity error constant

If $N \geq 2$, then $e_{ss} \rightarrow 0$ for a ramp input

Acceleration Input

$$R(s) = \frac{A}{s^3} \rightarrow E(s) = \frac{A}{s^2(1 + G_c G_p)}, \text{ so } e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{A}{s^2(1 + G_c G_p)} = \lim_{s \rightarrow 0} \frac{A}{s^2 G_c G_p}$$

Following a similar analysis as above, for a Type 2 system:

$$e_{ss} = \frac{A}{K_a}, \text{ where } K_a = \lim_{s \rightarrow 0} s^2 G_c G_p \leftarrow \text{Acceleration Error Constant}$$

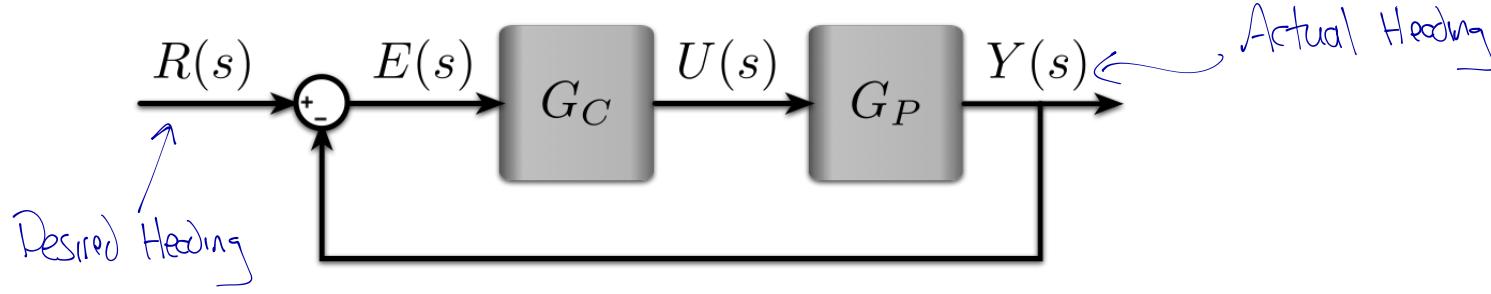
If $N \geq 3$, then $e_{ss} \rightarrow 0$ for accel. input

Steady-state Error of Systems (cont.)

Table 5.2 Summary of Steady-State Errors

Number of Integrations in $G_c(s)G(s)$, Type Number	Input		
	Step, $r(t) = A$, $R(s) = A/s$	Ramp, $r(t) = At$, $R(s) = A/s^2$	Parabola, $r(t) = At^2/2$, $R(s) = A/s^3$
0	$e_{ss} = \frac{A}{1 + K_p}$	∞	∞
1	$e_{ss} = 0$	$\frac{A}{K_v}$	∞
2	$e_{ss} = 0$	0	$\frac{A}{K_a}$

Example 5.3



$$\text{Define } G_P = \frac{K}{\tau s + 1}$$

Q: What can you say about this plant/vehicle?

$$G_C = k_p + \underbrace{\frac{k_I}{s}}$$

- Not oscillatory \rightarrow exponentially approaches desired state
- 1st-order system

Book uses K_I and K_D here. I'm using k_p and k_I because this is a PI controller
 k_p - proportional gain and k_I - integral gain

Q: What is the open-loop TF?

$$G_C G_P = \left[k_p + k_I/s \right] \left[\frac{K}{\tau s + 1} \right] = \left[\frac{k_p s + k_I}{s} \right] \left[\frac{K}{\tau s + 1} \right] = \frac{K(k_p s + k_I)}{s(\tau s + 1)}$$

Q: What is the closed-loop TF?

$$\frac{Y}{R} = \frac{G_C G_P}{1 + G_C G_P} = \frac{K(k_p s + k_I)}{s(\tau s + 1) + K(k_p s + k_I)} = \frac{K(k_p s + k_I)}{\tau s^2 + (1 + K k_p)s + K k_I}$$

Q: Without calculating the Final Value Theorem, what is the steady-state error to a step input?

$e_{ss} = 0$ because it is a Type 1 system (has 1 pure integrator)

Q: How about if $K_I = 0$? (Proportional Control only)

If $K_I = 0 \rightarrow G_C G_P = \frac{K k_p}{\tau s + 1} \rightarrow$ Type 0 system so

$$e_{ss} = \frac{A}{1 + B_p} \quad \text{where } B_p = \lim_{s \rightarrow 0} G_C G_P = K k_p$$

So...
 The integral term at the controller is needed to eliminate the steady-state error.

Example 5.3 (cont.)

Q: Would we want to issue step commands in desired heading?

Probably not... we know the system can't track that, for one.

So, let's look at a ramp input.

Q: What's the steady-state error to a ramp input?

$$G_c G_p = \frac{K(k_p s + k_I)}{s(\tau s + 1)}$$

The steady state error for ramp inputs is $e_{ss} = \frac{A}{K_U}$ where $K_U = \lim_{s \rightarrow 0} s G_c G_p$

$$K_U = \lim_{s \rightarrow 0} \frac{K(k_p s + k_I)}{(\tau s + 1)} = K k_I \quad \text{so} \quad e_{ss} = \frac{A}{K k_I}$$

Q: So, how should we choose k_I to minimize steady-state error?

high $k_I \rightarrow$ low e_{ss}

Q: Any problems with this?

Look at the closed-loop system

$$\frac{Y}{R} = \frac{K(k_p s + k_I)}{\tau s^2 + (1 + K k_p) s + K k_I} \quad \leftarrow \text{increasing } k_I \text{ increases } \omega_n$$

$$= \frac{2\sum \omega_n s + \omega_n^2}{s^2 + 2\sum \omega_n s + \omega_n^2} \quad \text{where } \omega_n^2 = \frac{K k_I}{\tau} \quad \text{and} \quad 2\sum \omega_n = \frac{1 + K k_p}{\tau}$$

$$\zeta = \frac{1 + K k_p}{2\sqrt{\omega_n}} \quad \text{so, all else fixed } \uparrow \omega_n \rightarrow \downarrow \zeta \text{ in this case}$$

Q: What does that mean for the response?

↑ overshoot

↑ oscillation / settling time

less ability to reject disturbance)

Performance Measures (Sec. 5.7)

Introduces some common-sense perf metrics and "optimum" systems for each. Please review.