

Example 2.2

Write the response of the dynamic system represented by:

$$\ddot{y} + 4\dot{y} + 3y = 2r$$

to initial conditions $y(0) = 1$, $\dot{y}(0) = 0$

and input $r(t) = 1 \text{ } \forall t \geq 0$

First, A short Aside

So far, when using Laplace TF we've assumed 0 initial cond.

We can look many transforms up in tables like the one in Table 2.3. Let's look closer at the $\frac{d^2}{dt^2}$ and $\frac{d}{dt}$ transforms

For function $f(t)$,

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

Q: How do we get that?

Remember that the Laplace Transform is defined as:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

Let's work from there

$$\mathcal{L}[f(t)] = \int_0^\infty (f(t)) e^{-st} dt = ? \quad \begin{array}{l} \text{Q: How do we integrate this equation?} \\ \text{Integration by parts} \end{array}$$

here: $u = f(t)$ and $dv = e^{-st} dt$

$$du = \frac{df}{dt} \quad v = -\frac{1}{s} e^{-st}$$

$$\int u dv = uv - \int v du$$

So

$$\mathcal{L}[f(t)] = \underbrace{f(t)}_u \left[\underbrace{-\frac{1}{s} e^{-st}}_v \right]_0^\infty - \int_0^\infty \underbrace{\left[\frac{df}{dt} \right]}_{du} \underbrace{\left[-\frac{1}{s} e^{-st} \right]}_v dt$$

Aside (cont.)

$$\mathcal{L}[f(t)] = f(0) \left[-\frac{1}{s} e^{-st} \right]_0^\infty - \int_0^\infty \left[\frac{df}{dt} \right] \left[-\frac{1}{s} e^{-st} \right] dt$$

Q: What is e^{-st} at $t=\infty$? at $t=0$?

$$= 0 \qquad \qquad \qquad = 1$$

So the first (definite integral) term becomes

$$f(\infty) 0 - f(0) \left[-\frac{1}{s} f(1) \right] = \frac{f(0)}{s}$$

Q: What does the $\int v du$ term look like?

$$\frac{1}{s} \int_0^\infty \left[\frac{df}{dt} \right] e^{-st} dt \leftarrow \text{This is just the Laplace TF of } \frac{df}{dt}!$$

So, the full expression becomes

$$\mathcal{L}[f(t)] = F(s) = \frac{f(0)}{s} + \frac{1}{s} \left[\mathcal{L}\left[\frac{df}{dt}\right] \right]$$

Rearranging terms, we see that

$$\boxed{\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)}$$

We could follow a similar path to find that

$$\boxed{\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2 F(s) - sf(0) - \dot{f}(0)}$$

Aside finished.

Back to the original problem...

Example 2.2

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Take the Laplace transform

$$[s^2Y(s) - sy(0)^1 - \dot{y}(0)^0] + 4[sY(s) - y(0)^1] + 3Y(s) = 2R(s)$$

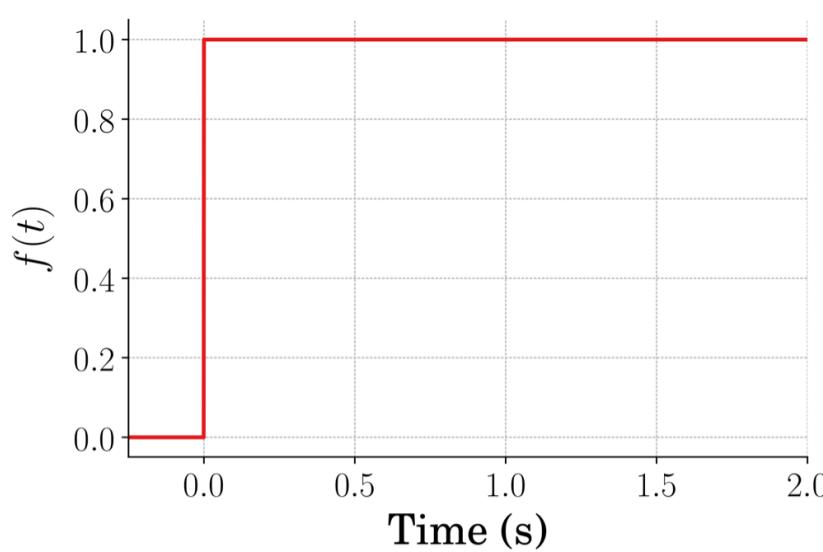
$$[s^2Y(s) - s] + 4[sY(s) - 1] + 3Y(s) = 2R(s)$$

$$s^2Y(s) - s + 4sY(s) - 4 + 3Y(s) = 2R(s)$$

$$[s^2 + 4s + 3]Y(s) - s - 4 = 2R(s)$$

Q: What is $R(s)$?

$r(t) = 1 \text{ } \forall t \geq 0 \leftarrow \text{This is a } \underline{\text{step function}}$



It will be one of the main types of commands that we use to analyze and characterize system dynamics (and therefore design controllers)

Do NOT fall into the trap of thinking this is the only kind of command we can issue a system.

The Laplace Transform of a step function is

$$\mathcal{L}[\text{step}] = \frac{1}{s} \leftarrow \text{You'll start to remember some of these. As you learn, look them up in Table 2.3.}$$

Example 2.2 (cont.)

$$[s^2 + 4s + 3]Y(s) - s - 4 = 2R(s)$$

r(t) is a step, so $R(s) = \frac{1}{s}$

$$[s^2 + 4s + 3]Y(s) - s - 4 = 2\left[\frac{1}{s}\right]$$

$$[s^2 + 4s + 3]Y(s) = s + 4 + 2\left[\frac{1}{s}\right]$$

$$Y(s) = \frac{s+4}{s^2 + 4s + 3} + \frac{2}{s(s^2 + 4s + 3)}$$

Q: How do we get back to the time domain?

Inverse Laplace Transform \leftarrow to use this (1) get a computer and/or

(2) manipulate the expression into a form available in Laplace TF tables

Let's do (2), at least this once.

To do so will use partial fraction expansion

$$\text{If } F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} \quad \text{for } m < n$$

and all p_i are unique (no repeated poles), we can write

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \dots + \frac{a_n}{s+p_n} \quad a_k \text{ are constants called residues at pole } p_k$$

To find a_k , solve

$$a_k = \left[(s+p_k) \frac{B(s)}{A(s)} \right]_{s=-p_k}$$

Example 2.2 (cont.)

$$y(s) = \frac{s+4}{s^2+4s+3} + \frac{2}{s(s^2+4s+3)} \quad \leftarrow \text{We're looking for the inverse Laplace of this}$$

We can look at the two components separately. Q: Why?: It's linear.

Looking at the 1st term

$$\frac{s+4}{s^2+4s+3} = \frac{s+4}{(s+1)(s+3)} = \frac{a_1}{s+1} + \frac{a_2}{s+3}$$

$$a_1 = \left[(s+p_1) \frac{s+4}{(s+1)(s+3)} \right]_{s=-p_1} \rightarrow p_1 = +1 \rightarrow \left[(s+1) \frac{s+4}{(s+1)(s+3)} \right]_{s=-1} = \left[\frac{s+4}{s+3} \right]_{s=-1} = \frac{3}{2}$$

$$a_2 = \left[(s+3) \frac{s+4}{(s+1)(s+3)} \right]_{s=-3} = \left[\frac{s+4}{s+1} \right]_{s=-3} = \frac{1}{-2}$$

$$\text{so } \frac{s+4}{s^2+4s+3} = \frac{3/2}{s+1} + \frac{(-1/2)}{s+3} = y_1(s) \quad \leftarrow \text{The mathematical linearity also means that superposition holds}$$

This component of the response is then

$$y_1(t) = \mathcal{L}^{-1} \left[\frac{3/2}{s+1} + \frac{(-1/2)}{s+3} \right] = \frac{3}{2} e^{-t} - \frac{1}{2} e^{-3t} \quad \leftarrow \text{Just look up the parts in the table}$$

Repeating this for the 2nd term in the equation, we find

$$y_2(s) = \frac{2}{s(s^2+4s+3)} = \frac{a_1}{s+1} + \frac{a_2}{s+3} + \frac{a_3}{s} = \frac{-1}{s+1} + \frac{1/3}{s+3} + \frac{2/3}{s}$$

$$y_2(t) = \mathcal{L}^{-1} \left[\frac{-1}{s+1} + \frac{1/3}{s+3} + \frac{2/3}{s} \right] = -e^{-t} + \frac{1}{3} e^{-3t} + \frac{2}{3}$$

So, the total response is

$$y(t) = y_1(t) + y_2(t) = \frac{3}{2} e^{-t} - \frac{1}{2} e^{-3t} - e^{-t} + \frac{1}{3} e^{-3t} + \frac{2}{3}$$

Example 2.2 (cont.)

$$y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} - e^{-t} + \frac{1}{3}e^{-3t} + 2/3$$

Q: What's the response as $t \rightarrow \infty$?

$$\lim_{t \rightarrow \infty} y(t) = 2/3 \quad \leftarrow \text{This is called the } \underline{\text{steady-state response}} \text{ of the system}$$

We can actually get to this directly from the system transfer function

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) \quad \leftarrow \underline{\text{Final Value Theorem}}$$

$$Y(s) = \frac{s+4}{s^2+4s+3} + \frac{2}{s(s^2+4s+3)} = \frac{s^2+4s+2}{s^3+4s^2+3s}$$

$$\lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \left[\frac{s^3+4s^2+2s}{s^3+4s^2+3s} \right] = \lim_{s \rightarrow 0} \left[\frac{s^2+4s+2}{s^2+4s+3} \right] = 2/3$$

Returning to ...

The State Differential Equation (Sec. 3.3)

Consider scalar $\dot{x}(t) = ax(t) + bu(t)$

Q: How would you solve this?

Just like the problem we just worked.

$$sX(s) - x(0) = aX(s) + bU(s)$$

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

If you take the inverse Laplace Transform of this, you get

$$x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)} b u(\tau) d\tau$$

The same form holds for the matrix version

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

If we define the matrix exponential:

$$e^{At} = \exp(At) = I + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^k t^k}{k!} + \dots$$

then the solution to the state-space equation is

$$\bar{x}(t) = \exp(At)\bar{x}(0) + \int_0^t \exp[A(t-\tau)] B \bar{u}(\tau) d\tau$$

Returning to ...

The State Differential Equation (cont.)

$$\dot{\bar{X}} = A\bar{X} + B\bar{U}$$

Take the Laplace Transform of this

$$s\bar{X}(s) - \bar{X}(0) = A\bar{X}(s) + B\bar{U}(s)$$

$$(sI - A)\bar{X}(s) = \bar{X}(0) + B\bar{U}(s)$$

$I = \text{identity matrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix}$
ones on diagonal,
zeros elsewhere

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}Bu(s)$$

Defn $\Phi(s) = (sI - A)^{-1}$

This is the Laplace Transform of

$$\tilde{\Phi}(t) = \exp(At)$$

← This is the state transition matrix

It describes the free response of the system

no input

In the free response case (only initial conditions, no input)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \dots & \phi_{1n}(t) \\ \phi_{21}(t) & \dots & \phi_{2n}(t) \\ \vdots & \ddots & \vdots \\ \phi_{n1}(t) & \dots & \phi_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

To find $\phi_{ij}(t)$ zero all initial cond except $x_j(0)$ and observe the response of state x_i . Repeat for all pairings.